Midterm #1 (20% + 5% Bonus Points)

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Nov. 3rd, 2014

Any academic dishonesty will automatically lead to zero point.

1) (2%) Prove that $\sqrt{p}$ is irrational for any prime p.

   Solution:

   If $\sqrt{p}$ is not irrational for any prime p, we have $\sqrt{p} = \frac{a}{b}$, where $a, b \in \mathbf{Z}^+$ and $gcd(a, b) = 1$. Then $\sqrt{p} = \frac{a}{b} \Rightarrow p = \frac{a^2}{b^2} \Rightarrow pb^2 = a^2 \Rightarrow p|a^2 \Rightarrow p|a$. We know that $a = pk \exists k \in \mathbf{Z}^+$, since $p|a$. Besides, $pb^2 = a^2 = (pk)^2$, or $b^2 = pk^2$. Hence, $p|b^2 \Rightarrow p|b$. However, if $p|a$ and $p|b$ then $gcd(a, b) = p > 1$. Here we can see that our conclusion $gcd(a, b) = p > 1$ contradicts our assumption $gcd(a, b) = 1$ in the very beginning.

2) (6%) Lucas numbers are closely related to Fibonacci numbers. This sequence is defined recursively as follow:

   a) $L_0 = 2, L_1 = 1$
   b) $L_{n+2} = L_{n+1} + L_n$ for $n \geq 0$

   The following proofs should be described in detail to demonstrate your understanding of the Principle of Mathematical Induction

   a) Prove that $L_1^2 + L_1^2 + L_3^2 + \ldots + L_n^2 = L_n L_{n+1} - 2, \ \forall n \in \mathbf{N}$
   b) Prove that $5F_{n+2} = L_{n+4} - L_n, \ \forall n \in \mathbf{N}$, where F_n denotes the n^{th} Fibonacci number

   Solution:

   a) We adopt the of Principle of Mathematical Induction to prove $L_1^2 + L_1^2 + L_3^2 + \ldots + L_n^2 = L_n L_{n+1} - 2$.

      For n=1, we find $L_1^2 = 1^2 = 1 = 1 \times 3 - 2 = L_1 L_2 - 2$

      We assume the result is true when $n = k$. This gives us $\sum_{i=1}^{k} L_i^2 = L_k L_{k+1} - 2$. Next, for $n = k + 1$ we find that $\sum_{i=1}^{k+1} L_i^2 = \sum_{i=1}^{k} L_i^2 \ + \ L_{k+1}^2 = L_k L_{k+1} - 2 + L_{k+1}^2 = L_{k+1}(L_k + L_{k+1}) - 2 = L_{k+1} L_{k+2} - 2$

b) We adopt the Alternative Form of Principle of Mathematical Induction to prove $5F_{n+2} = L_{n+4} - L_n$.
For $n = 0$, $5F_{0+2} = 5F_2 = 5(1) = 5 = 7 - 2 = L_4 - L_0 = L_{0+4} = L_0$
For $n = 1$, $5F_{1+2} = 5F_3 = 5(2) = 10 = 11 - 1 = L_5 - L_1 = L_{1+4} = L_1$
Next, we assume the induction hypothesis – that is, that for some $k \geq 1$, $5F_{n+2} = L_{n+4} - L_n$ for all $n = 0, 1, 2, \ldots, k-1, k$. It then follows that for $n = k+1$, $5F_{(k+1)+2} = 5F_{k+3} = 5(F_{k+2} + F_{k+1}) = 5(F_{k+2} + F_{(k-1)+2}) = 5F_{k+2} + 5F_{(k-1)+2} = (L_{k+4} - L_k) + (L_{(k-1)+4} - L_{(k-1)}) = (L_{k+4} - L_k) + (L_{k+3} - L_{k-1}) = (L_{k+4} + L_{k+3}) - (L_k + L_{k-1}) = L_{k+5} - L_{k+1} = L_{(k+1)+4} + L_{k+1}$. Hence, it then follows that $\forall n \in \mathbf{N}\ 5F_{n+2} = L_{n+4} - L_n$.

3) (3%) Use the law of set theory to simplify each of the following:
Hint:(a), (b) and (c) should be simplified to denote as a single set, such as A or B

a) $A \cap (B - A)$
b) $(A \cap B) \cup (A \cap B \cap \overline{C} \cap D) \cup (\overline{A} \cap B)$
c) $(A - B) \cup (A \cap B)$

Solution:

a) $A \cap (B - A) = A \cap (B \cap \overline{A}) = B \cap (A \cap \overline{A}) = \emptyset$
b) $(A \cap B) \cup (A \cap B \cap \overline{C} \cap D) \cup (\overline{A} \cap B) = (A \cap B) \cup (\overline{A} \cap B) = (A \cup \overline{A}) \cap B = B$
c) $(A - B) \cup (A \cap B) = (A \cap \overline{B}) \cup (A \cap B) = A \cap (\overline{B} \cup B) = A$

4) (2 %) For $A = \{1, 2, 3, 4, 5, 6, 7\}$, determine the number of

a) subsets of A
b) nonempty proper subsets of A
c) subsets of A containing four elements
d) subsets of A containing 1 and 2
e) subsets of A containing five elements, including 1 and 2
f) subsets of A containing with an even number of elements

Solution:

a) 2^7
b) 126
c) $\binom{7}{4}$

d) 2^5

e) $\binom{5}{3}$

f) $\binom{7}{0} + \binom{7}{2} + \binom{7}{4} + \binom{7}{6} = 64$

5) (3%) For primitive statements p, q:

a) Verify that $p \to [q \to (p \land q)]$ is a tautology.

b) Verify that $(p \lor q) \to [q \to q]$ is a tautology by using the results from part (a) along with the substitution rules and the laws of logic.

c) Is $(p \lor q) \to [q \to (p \land q)]$ a tautology?

Solution:

a)

p	q	$p \land q$	$q \to (p \land q)$	$p \to [q \to (p \land q)]$
0	0	0	1	1
0	1	1	0	0
1	0	1	1	1
1	1	1	1	1

b) Replace p to $(p \land q)$ in (a)

$\Rightarrow (p \land q) \to [q \to ((p \land q) \land q)]$ (the substitution rules)

$\Rightarrow (p \land q) \to [q \to q]$ (the absorption law)

$\Rightarrow p \to [q \to (p \land q)] \Leftrightarrow (p \land q) \to [q \to q]$ is a tautology

c)

p	q	$p \lor q$	$q \to (p \land q)$	$(p \land q) \to [q \to (p \land q)]$
0	0	0	1	1
0	1	1	0	0
1	0	1	1	1
0	1	1	1	1

6) (2%) Establish the validity of the following argument. You need to give reasons for the steps in your verification of the validity

$(\neg p \vee q) \rightarrow r$

$r \rightarrow (s \vee t)$

$\neg s \wedge \neg u$

$\neg u \rightarrow \neg t$

$\therefore\ p$

Solution:

a) $\neg s \vee \neg u$ Premise
b) $\neg u$ Step (a) and the Rule of Conjunctive Simplification
c) $\neg u \rightarrow \neg t$ Premise
d) $\neg t$ Steps (b), (c) and the Rule of Detachment
e) $\neg s$ Step (a) and the Rule of Conjunctive Simplification
f) $\neg s \vee \neg t$ Steps (d), (e) and the Rule of Conjunction
g) $r \rightarrow (s \vee t)$ Premise
h) $\neg(s \vee t) \rightarrow \neg r$ Step (g) and $[r \rightarrow (s \vee t)] \leftrightarrow [\neg(s \vee t) \rightarrow r]$
i) $(\neg s \wedge \neg t) \rightarrow \neg r$ Step (h) and DeMorgan's Laws
j) $\neg r$ Steps (f), (i), and the Rule of Detachment
k) $(\neg p \vee q) \rightarrow r$ Premise
l) $\neg r \rightarrow \neg(\neg p \vee q)$ Step (k) and $[(\neg p \vee q) \rightarrow] \leftrightarrow [\neg r \rightarrow \neg(\neg p \vee q)]$
m) $\neg r \rightarrow (p \wedge \neg q)$ Step (l) and DeMorgan's Laws and the Law of Double Negation
n) $p \wedge \neg q$ Steps (j), (m) and Rule of Detachment
o) $\therefore p$ Step (n) and the Rule of Conjunctive Simplification

7) (3%) Answer the following questions:

a) Determine the coefficient of $w^3x^2yz^2$ in $(2w - x + 3y - 2z)^8$
b) Determine the coefficient of $w^2x^2y^2z^2$ in $(v + w - 2x + y + 5z + 3)^{12}$
c) Determine x if $\sum_{i=0}^{50} \binom{50}{i} 8^i = x^{100}$

Solution:

a) $\binom{8}{3,2,1,2}(2)^3(-1)^2(3)(-2)^2$

b) $\binom{12}{0,2,2,2,2,4}(1)^2(-2)^2(1)^2(5)^2(3)^4 = \frac{12!}{2!^4\,4!} \times (2)^2(5)^2(3)^4$

c) $\sum_{i=50}^{50}\binom{50}{i}8^i = (1+8)^{50} = (9)^{50} = \left[(\pm 3)^2\right]^{50} = (\pm 3)^{100}$, so $x = \pm 3$

8) (3%) Answer the following questions:

a) In how many possible ways could a student answer a 10-question true-false test?

b) In how many ways can the student answer the above test if he/she can leave each question unanswered to avoid extra penalty for wrong answers?

c) In how many ways can the student answer the above test if he/she can leave exactly one question unanswered to avoid extra penalty for wrong answers?

Solution:

a) With two choices per question, there are 2^{10} ways.

b) With three choices per question, there are 3^{10} ways.

c) The student selects the question to be unanswered our of 10 questions and answer the rest of them. $\binom{10}{1}2^9 \times 3$