Name: Student ID:

Quiz #7 6%

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This is a closed book test. Any academic dishonesty will automatically lead to zero point.

1) (1%) Suppose that p_1, p_2, and p_3 are distinct primes and that $n, k \in \mathbf{Z}^+$ with $n = p_1^5 p_2^3 p_3^k$. Let A be the set of positive integer divisors of n and define the relation $\mathscr{R}$ on A by $x \mathscr{R} y$ if x exactly devides y. If there are 5880 ordered pairs in $\mathscr{R}$ determine k and $|A|$

   Solution: Since $5880 = \binom{6+2-1}{2}\binom{4+2-1}{2}\binom{(k-1)+2-1}{2} = \binom{7}{2}\binom{5}{2}\binom{k+2}{2} = (21)(10)\frac{1}{2}(k+2)(k+1)$, we find that $56 = (k+2)(k+1)$ and $k = 6$

   For $n = p_1^5 p_2^3 p_3^k$ there are $(5+1)(3+1)(6+1) = 168$ positive integer divisors, so $|A| = 168$.

2) (1%) With $A = \{1, 2, 3, 4\}$, let $\mathscr{R} = \{(1, 1), (1, 2), (2, 3), (3, 3), (3, 4), (4, 4)\}$ be a relation on A. Find two relations $\mathscr{S}, \mathscr{T}$ on A where $\mathscr{S} \neq \mathscr{T}$ but $\mathscr{R} \circ \mathscr{S} = \mathscr{R} \circ \mathscr{T} = \{(1, 1), (1, 2), (1, 4)\}$.

   Solution: Let $\mathscr{S} = \{(1, 1), (1, 2), (1, 4)\}$ and $\mathscr{T} = \{(2, 1), (2, 2), (1, 4)\}$.

3) (2%) Define the relation $\mathscr{R}$ on the set $\mathbf{Z}$ by $a\mathscr{R}b$ if $a - b$ is a nonnegative even integer. Verify that $\mathscr{R}$ defines a partial order for $\mathbf{Z}$. Is this a partial order or total order?

   Solution: For each $a \in \mathbf{Z}$, it follows that $a\mathscr{R}a$ because $a - a = 0$. 0 is definitely an even nonnegative integer. Hence, $\mathscr{R}$ is *reflexive*. If $a, b, c \in \mathbf{Z}$ with $a\mathscr{R}b$ and $b\mathscr{R}c$, then $a - b = 2m$ for some $m \in \mathbf{N}$ and $b - c = 2n$ for some $n \in \mathbf{N}$. We can conclude $a\mathscr{R}c$ because $a - c = (a - b) + (b - c) = 2(m + n)$ where $m + n \in \mathbf{N}$. Therefore, $\mathscr{R}$ is *transitive*. Finally, suppose that $a\mathscr{R}b$ and $b\mathscr{R}a$ for some $a, b \in \mathbf{Z}$. Then $a - b$ and $b - a$ are both nonnegative integers. Since this can only occur when $a - b = b - a$, we find that a must equal to b. Hence, $\mathscr{R}$ is *antisymmetric* since $a\mathscr{R}b \wedge b\mathscr{R}a \Rightarrow a = b$. Consequently, the relation $\mathscr{R}$ is a partial order for $\mathbf{Z}$. However, it is not a total order. For example, $2, 3 \in \mathbf{Z}$ and we have neither $2\mathscr{R}3$ nor $3\mathscr{R}2$, because neither -1 nor 1 is a nonnegative even integer, respectively.

4) (2%) Let $A = \{1, 2, 3, 4, 5, 6, 7\}$. We define $\mathscr{R}$ on A by $(x, y) \in \mathscr{R}$ if $x - y$ is a multiple of 3.

a) Show that $\mathscr{R}$ is an equivalence relation on A.

b) Determine the equivalence classes and partition of A induced by $\mathscr{R}$.

Solution:

a) For all $a \in A$, $a - a = 3 \cdot 0$. So, $\mathscr{R}$ is reflexive. If $a, b \in A$, then $a - b = 3c$ for some $c \in \mathbf{Z} \;\Rightarrow\; b - a = 3(-c)$ for some $-c \in \mathbf{Z}$. Hence, $\mathscr{R}$ is symmetric, since $a\mathscr{R}b \Rightarrow b\mathscr{R}a$. If $a, b, c \in A$ and $a\mathscr{R}b$, $b\mathscr{R}c$, then $a - b = 3m, b - c = 3n$ for some $m, n \in \mathbf{Z} \;\Rightarrow (a - b) + (b - c) = a - c = 3(m + n) \Rightarrow a\mathscr{R}c$. Consequently, $\mathscr{R}$ is transitive.

b) $[1] = [4] = [7] = \{1, 4, 7\}$; $[2] = [5] = \{2, 5\}$; $[3] = [6] = \{3, 6\}$. $A = \{1, 4, 7\} \cap \{2, 5\} \cap \{3, 6\}$.