SOLUTION

Ex 11.1: 2, 3, 5, 8, 13

Ex 11.2: 1, 2, 3, 9, 15

Ex 11.3: 3, 4, 5, 20, 23

Ex 11.4: 2, 3, 13, 14, 26

Ex 11.5: 1, 3, 4, 6, 19

Ex 11.6: 1, 2, 5, 6, 13

Ex 11.1: (2)

a) $\{b,e\},\{e,f\},\{f,g\},\{g,e\},\{e,b\},\{b,c\},\{c,d\}$
b) $\{b,e\},\{e,f\},\{f,g\},\{g,e\},\{e,d\}$
c) $\{b,e\},\{e,d\}$
d) $\{b,e\},\{e,f\},\{f,g\},\{g,e\},\{e,b\}$
e) $\{b,e\},\{e,f\},\{f,g\},\{g,e\},\{e,d\},\{d,c\},\{c,b\}$
f) $\{b,a\},\{a,c\},\{c,b\}$

Ex 11.1: (3)

- 6

Ex 11.1: (5)

- Each path from a to h must include the edge $\{b, g\}$. There are three paths (in G) from a to b and three paths (in G) from g to h. Consequently, there are nine paths from a to h in G.
- There is only one path of length 3, two of length 4, three of length 5, two of length 6, and on of length 7.

Ex 11.1: (8)

- The smallest number of guards needed is 3 – e.g., at vertices a, g, i.

Ex 11.1: (13)

- This relation is reflexive, symmetric and transitive, so it is an equivalence relation. The partition of V induced by R yields the (connected) components of G.

Ex 11.2: (1)

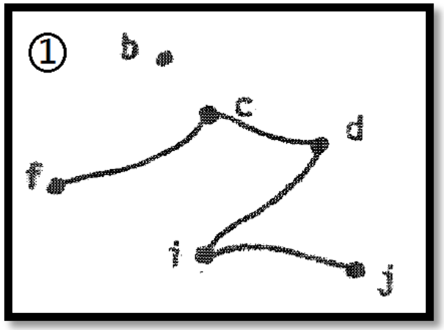
a) Three:
(1) $\{b, a\}, \{a, c\}, \{c, d\}, \{d, a\}$
(2) $\{f, c\}, \{c, a\}, \{a, d\}, \{d, c\}$
(3) $\{i, d\}, \{d, c\}, \{c, a\}, \{a, d\}$

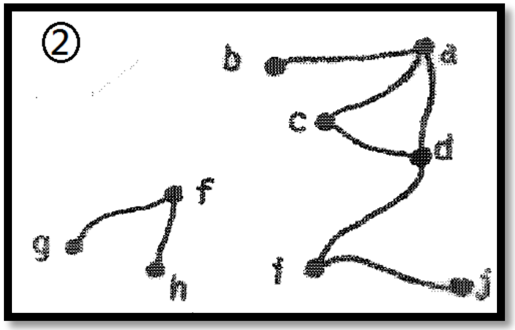
b) G_1 is the subgraph induced by $U = \{a, b, d, f, g, h, i, j\}$.
$G_1 = G - \{c\}$.

c) G_2 is the subgraph induced by $W = \{b, c, d, f, g, i, j\}$.
$G_2 = G - \{a, h\}$.

d) Fig.(1).

e) Fig.(2).





Ex 11.2: (2)

a) G_1 is not an induced subgraph of G if there exists an edge $\{a, b\}$ in E such that $a, b \in V$, but $\{a, b\} \notin E_1$.

b) Let $e = \{a, d\}$. Then $G - e$ is a subgraph of G but it is not an induced subgraph.

Ex 11.2: (3)

a) There are $2^9 = 512$ spanning subgraphs.
b) Four of the spanning subgraph in part (a) are connected.
c) 2^6

Ex 11.2: (9)

a) Each graph has four vertices that are incident with three edges. In the second graph these vertices (w, x, y, z) form a cycle. This is not so for the corresponding vertices (a, b, g, h) in the first graph. Hence the graphs are not isomorphic.

b) In the first graph the vertex d is incident with four edges, No vertex in the second graph has this property, so the graphs are not isomorphic.

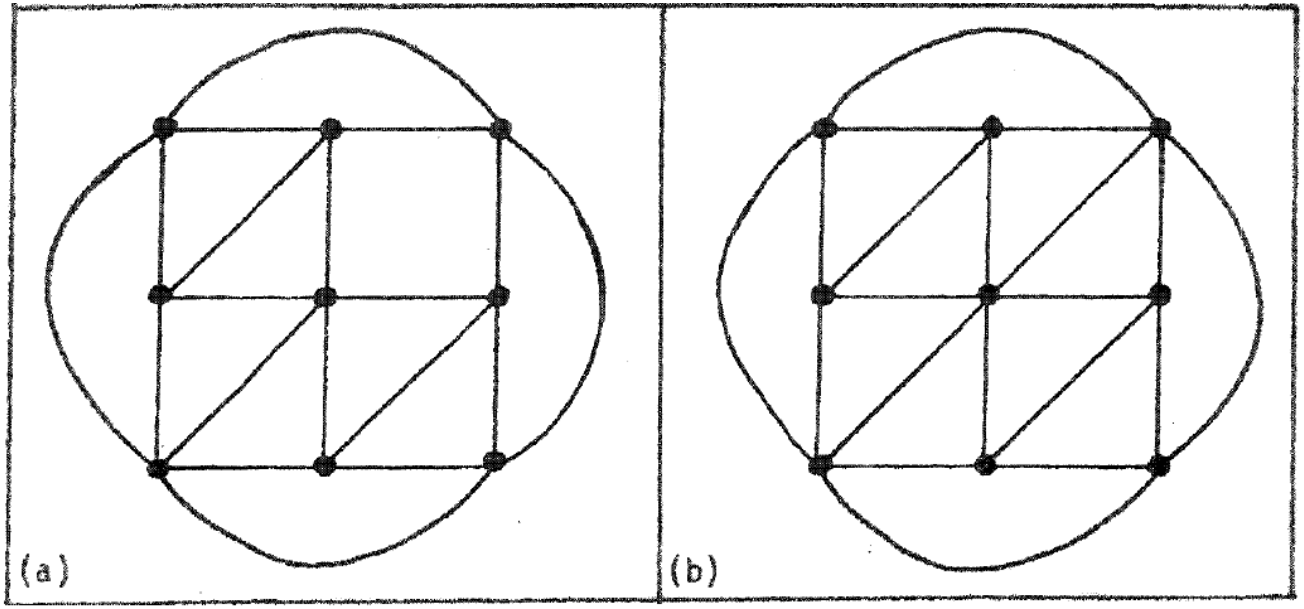
Ex 11.2: (15)

a) Here f must also maintain directions. So if $(a, b) \in \mathrm{E}_1$, then $(f(a), f(b)) \in E_2$.

b) They are not isomorphic. Consider vertex a in the first graph. It is incident to one vertex and incident from two other vertices. No vertex in the other graph has this property.

Ex 11.3: (3)

- Since $38 = 2|E| = \sum_{v \in V} \deg(v) \geq 4|V|$, the largest possible value for $|V|$ is 9. We can have (i) seven vertices of degree 4 and two of degree 5; or (ii) eight vertices of degree 4 and one of degree 6. The graph in part (a) of the figure is an example for case (i); an example for case (ii) is provided in part (b) of the figure.



Ex 11.3: (4)

a) We must note here that G need not be connected. Up to isomorphism G is either a cycle on six vertices or (a disjoint union of) two cycles, each on three vertices.
b) Here G is either a cycle on seven vertices or (a disjoint union of) two cycles – one on three vertices and the other on four.
c) For such a graph G_1, $\overline{G_1}$ is one of the graphs in part (a). Hence there are two such graphs G_1.
d) Here $\overline{G_1}$ is one of the graphs in part (b). There are two such graphs G_1(up to isomorphism).
e) Let $G_1 = (V_1, E_1)$be a loop-free undirected $(n-3)$-regular graph with $|V| = n$. Up to isomorphism the number of such graphs G_1 is the number of partitions of n into summands that exceed 2.

Ex 11.3: (5)

a) $|V_1| = 8 = |V_2|$; $|E_1| = 14 = |E_2|$.

b) For V_1 we find that $\deg(a) = 3, \deg(b) = 4, \deg(d) = 3$, $\deg(e) = 3, \deg(f) = 4, \deg(g) = 4$, and $\deg(h) = 3$. For V_2 we have $\deg(s) = 3, \deg(t) = 4, \deg(u) = 4, \deg(v) = 3, \deg(w) = 4, \deg(x) = 3, \deg(y) = 3$, and $\deg(z) = 4$. Hence each of the two graphs has four vertices of degree 3 and four of degree 4.

c) Despite the results in parts (a) and (b) the graphs G_1 and G_2 are not isomorphic.

In the graph G_2 the four vertices of degree 4 – namely, t, u, w, and z – are on a cycle of length 4. For the graph G_1 the vertices b, c, f, and g – each of degree 4 – do not lie on a cycle of length 4.

A second way to observe that G_1 and G_2 are not isomorphic is to consider once again the vertices of degree 4 in each graph. In G_1 these vertices induce a disconnected subgraph consisting of the two edges $\{b, c\}$and $\{f, g\}$. The four vertices of degree 4 in graph G_2 induce a connected subgraph that has five edges – every possible edge except $\{u, z\}$.

Ex 11.3: (20)

a) $a \to b \to c \to g \to k \to j \to g \to b \to f \to j \to i \to f \to e \to i \to h \to d \to e \to b \to d \to a.$

b) $d \to a \to b \to d \to g \to i \to e \to f \to i \to j \to f \to b \to c \to g \to k \to j \to g \to b \to e.$

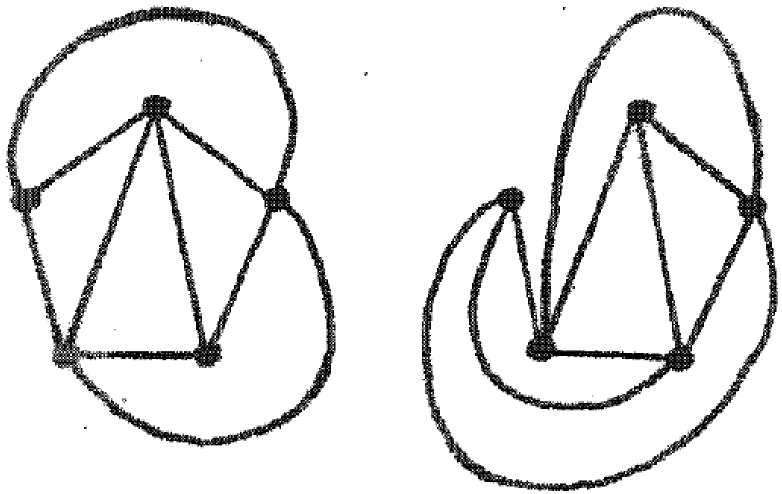
Ex 11.3: (23)

- Yes. Model the situation with a graph where there is a vertex for each room and the surrounding corridor. Draw an edge between two vertices if there is a door common to both rooms, or a room and the surrounding corridor. The resulting multigraph is connected with every vertex of even degree.

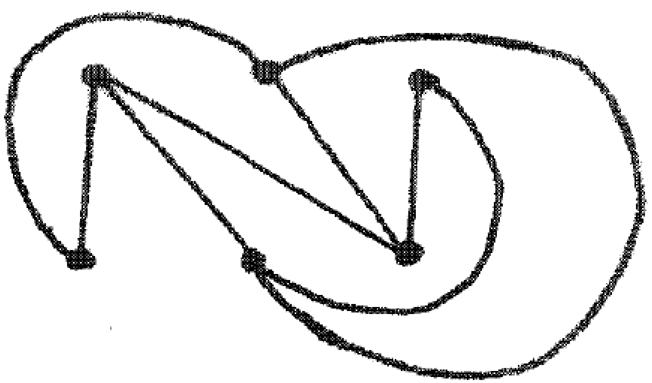
Ex 11.4: (2)

- From the symmetry in these graphs the following demonstrate the situations we must consider

K_5:



$K_{3,3}$:



Ex 11.4: (3)

a)

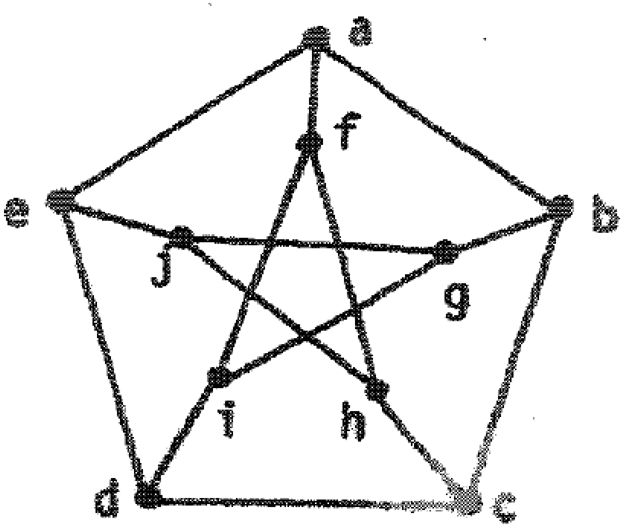
Graph	Number of vertices	Number of edges
$K_{4,7}$	11	28
$K_{7,11}$	18	77
$K_{m,n}$	$m+n$	mn

b) $m = 6$

Ex 11.4: (13)

a)

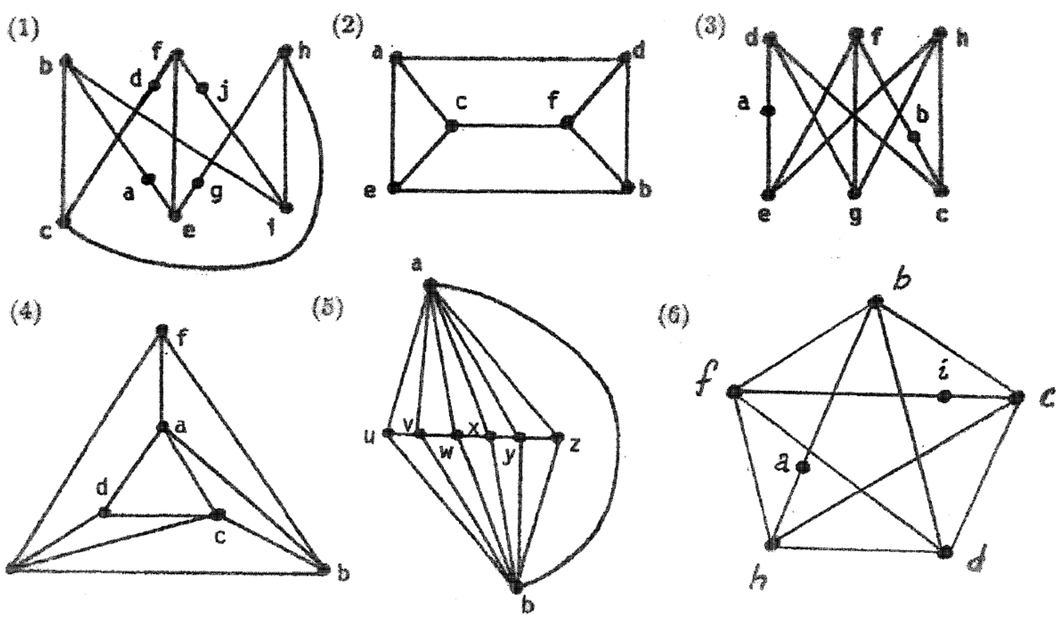
a:	{1,2},	f:	{4,5},
b:	{3,4},	g:	{2,5},
c:	{1,5},	h:	{2,3},
d:	{2,4},	i:	{1,3},
e:	{3,5},	j:	{1,4}.



b) G is (isomorphic to) the Petersen graph. (See Fig. 11.52(a)).

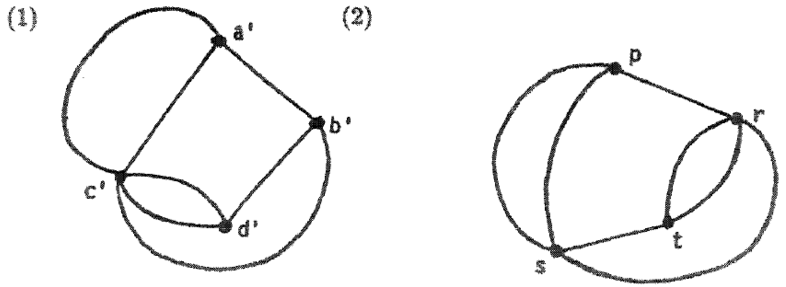
Ex 11.4: (14)

- Graph (1) shows that the first graph contains a subgraph homeomorphic to $K_{3,3}$, so it is not planar. The second graph is planar and isomorphic to the second graph of the exercise. The third graph provides a subgraph homeomorphic to $K_{3,3}$ so the third graph given here is not planar. Graph (6) is not planar because it contains a subgraph homeomorphic to K_5.

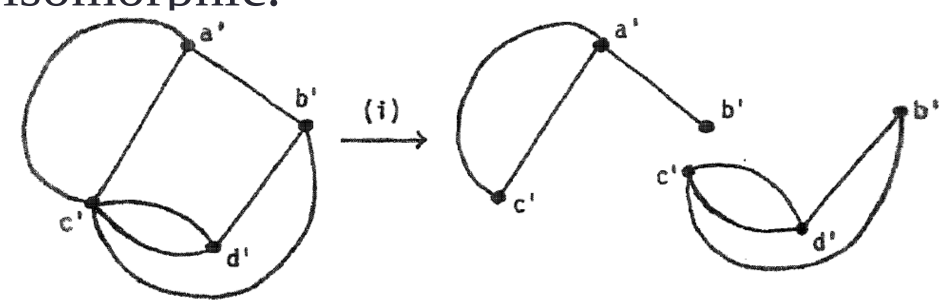


Ex 11.4: (26)

a) The correspondence $a \to v, b \to w, c \to y, d \to z, e \to x$ provides an isomorphism.

b) 

c) In the first graph in part (b) vertex c' had degree 5. Since no vertex had degree 5 in the second graph, the two graphs cannot be isomorphic.

d) 

e) $\{\{a', c'\}, \{c', b'\}, \{b', a'\}\}$; $\{\{p, r\}, \{r, t\}, \{r, t\}, \{r, s\}\}$.

Ex 11.5: (1)

(a) (b) (c) (d)

Ex 11.5: (3.a~3.d)

a) Hamilton cycle: $a \to g \to k \to i \to h \to b \to c \to d \to j \to f \to e \to a$
b) Hamilton cycle: $a \to d \to b \to e \to g \to j \to i \to f \to h \to c \to a$
c) Hamilton cycle: $a \to h \to e \to f \to g \to i \to d \to c \to b \to a$
d) The edges $\{a, c\}, \{c, d\}, \{d, b\}, \{b, e\}, \{e, f\}, \{f, g\}$ provide a Hamilton path for the given graph. However, there is no Hamilton cycle, for such a cycle would have to include the edges $\{b, d\}, \{b, e\}, \{a, c\}, \{a, e\}, \{g, f\}$, and $\{g, e\}$ – and, consequently, the vertex e will have degree greater than 2.

Ex 11.5: (3.e, 3.f)

e) The path $a \rightarrow b \rightarrow c \rightarrow d \rightarrow e \rightarrow j \rightarrow i \rightarrow h \rightarrow g \rightarrow f \rightarrow k \rightarrow l \rightarrow m \rightarrow n \rightarrow o$ is one possible Hamilton path for this graph. Another possibility is the path $a \rightarrow b \rightarrow c \rightarrow d \rightarrow i \rightarrow h \rightarrow g \rightarrow f \rightarrow k \rightarrow l \rightarrow m \rightarrow n \rightarrow o \rightarrow j \rightarrow e$. However, there is no Hamilton cycle. For if we try to construct a Hamilton cycle we must include the edges $\{a, b\}, \{a, f\}, \{f, k\}, \{k, l\}, \{d, e\}, \{e, j\}, \{j, o\}$ and $\{n, o\}$. This then forces us to eliminate the edges $\{f, g\}$ and $\{i, j\}$ from further consideration . Now consider the vertex i,. If we use edges $\{d, i\}$ and $\{i, n\}$, then we have a cycle on the vertices d, e, j, o, n and i – and we cannot get a Hamilton cycle for the given graph. Hence we must use only one of the edges $\{d, i\}$ and $\{i, n\}$. Because of the symmetry in this graph let us select edge $\{d, i\}$ – and then edge $\{h, i\}$ so that vertex i will have degree 2 in the Hamilton cycle we are trying to construct. Since edges $\{d, i\}$ and $\{d, e\}$ are now being used, we eliminate edge $\{c, d\}$ and this then forces us to include edges $\{b, c\}$ and $\{e, h\}$ in our construction. Also we must include the edge $\{m, n\}$ since we eliminated edge $\{i, n\}$ from consideration. Next we eliminate edge $\{l, g\}$. But now we have eliminated the four edges $\{b, g\}, \{f, g\}, \{h, g\}$ and $\{l, g\}$ and g is consequently isolated.

f) For this graph we find the Hamilton cycle $a \rightarrow b \rightarrow c \rightarrow d \rightarrow e \rightarrow j \rightarrow i \rightarrow h \rightarrow g \rightarrow l \rightarrow m \rightarrow n \rightarrow o \rightarrow t \rightarrow s \rightarrow r \rightarrow q \rightarrow p \rightarrow k \rightarrow f \rightarrow a$.

Ex 11.5: (4)

a) Consider the graph as shown in Fig.11.52(a). We demonstrate one case. Start at vertex a and consider the partial path $a \to f \to i \to d$. These choices require the removal of edge {f,h} and {g,i} from further consideration since each vertex of the graph will be incident with exactly two edges in the Hamilton cycle. At vertex d we can go to either vertex c or vertex e. (i) If we go to vertex c we eliminate edge {e,d} from consideration, but we must now incude edges {e,j} and {e,a}, and this forces the elimination of edge {a,b}. Now we must consider vertex b, for by eliminating edge {a,b}. We are now required to include edges {b,g} and {b,c} in the cycle. This forces us to remove edge {c,h} from further consideration. But we have now removed edges {f,h} and {c,h} and there is only one other edge that is incident with h, so no Hamilton cycle can be obtained. (ii) Selecting vertex e after d, we remove edge {d,c} and include {c,h} and {b,c}. Having removed {g,i} we must include {g,b} and {g,j}. This forces the elimination of {a,b}, the inclusion of {a,e} (and the elimination of {e,j}). We now have a cycle containing a, f, i, d, e, hence this method has also failed.
However, this graph does have a Hamilton path: $a \to b \to c \to d \to e \to j \to h \to f \to i \to g$.

b) For example, remove vertex j and the edges {e,j}, {g,j}, {h,j}. Then $e \to a \to f \to h \to c \to b \to g \to i \to d \to e$ provides a Hamilton cycle for this subgraph.

Ex 11.5: (6)

- Let the vertices on the cycle (rim) of W_n be consecutively denoted by $v_1, v_2, \ldots, v_n$, and let v_{n+1} denote the additional (central) vertex of W_n. Then the following cycles provide n Hamilton cycles for the wheel graph W_n.

(1) $v_1 \rightarrow v_{n+1} \rightarrow v_2 \rightarrow v_3 \rightarrow v_4 \rightarrow \cdots \rightarrow v_{n-1} \rightarrow v_n \rightarrow v_1$;

(2) $v_1 \rightarrow v_2 \rightarrow v_{n+1} \rightarrow v_3 \rightarrow v_4 \rightarrow \cdots \rightarrow v_{n-1} \rightarrow v_n \rightarrow v_1$;

(3) $v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow v_{n+1} \rightarrow v_4 \rightarrow \cdots \rightarrow v_{n-1} \rightarrow v_n \rightarrow v_1$;

…

(n-1) $v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow v_4 \rightarrow \cdots \rightarrow v_{n-1} \rightarrow v_{n+1} \rightarrow v_n \rightarrow v_1$;

(n) $v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow v_4 \rightarrow \cdots \rightarrow v_{n-1} \rightarrow v_n \rightarrow v_{n+1} \rightarrow v_1$;

Ex 11.5: (19)

- This follows from Theorem 11.9, since for all (nonadjacnet) $x, y \in V, \deg(x) + \deg(y) = 12 > 11 = |V|$.

Ex 11.6: (1)

- Draw a vertex for each species of fish. If two species x, y must be kept in separate aquaria, draw the edge $\{x, y\}$. The smallest number of aquaria needed is then the chromatic number if the resulting graph.

Ex 11.6: (2)

- Draw a vertex for each committee. If someone serves on two committees c_i, c_j draw the edge joining the vertices for c_i and c_j. Then the least number of meeting times is the chromatic number of the graph.

Ex 11.6: (5)

a) $P(G, \lambda) = \lambda(\lambda - 1)^3$.

b) For $G = K_{1,n}$ we find that $P(G, \lambda) = \lambda(\lambda - 1)^n$. $\chi(K_{1,n}) = 2$.

Ex 11.6: (6)

a) (i) Here we have λ choices for vertex a, 1 choice for vertex b (the same choice as that for vertex a), and$\lambda - 1$ choices for each of vertices x, y, z. Consequently, there are $\lambda(\lambda-1)^3$proper colorings of $K_{2,3}$ where vertices a and b are colored the same.
(ii) Now we have λ choices for vertex a, $\lambda - 1$ choices for vertex b, and $\lambda - 2$ choices for each of the vertices x, y, and z. And here there are $\lambda(\lambda-1)(\lambda-2)^3$ proper colorings.

b) Since the two cases in part (a) are exhaustive and mutually exclusive, the chromatic polynomial for $K_{2,3}$ is
$\lambda(\lambda-1)^3 + \lambda(\lambda-1)(\lambda-2)^3$
$= \lambda(\lambda-1)(\lambda^3 - 5\lambda^2 + 10\lambda - 7)$. $\chi(K_{2,3}) = 2$.

c) $P(K_{2,n}, \lambda) = \lambda(\lambda-1)^n + \lambda(\lambda-1)(\lambda-2)^n$. $\chi(K_{2,n}) = 2$.

Ex 11.6: (13)

a) $|V| = 2n$; $|E| = \left(\frac{1}{2}\right)\sum_{v \in V} \deg(v) = \left(\frac{1}{2}\right)[4(2) + (2n-4)(3)] = \left(\frac{1}{2}\right)[8 + 6n - 12] = 3n - 2, n \geq 1.$

b) For $n = 1$, we find that $G = K_2$ and $P(G, \lambda) = \lambda(\lambda - 1) = \lambda(\lambda - 1)(\lambda^2 - 3\lambda + 3)^{1-1}$so the result is true in this first case. For $n = 2$, we have $G = C_4$, the cycle of length 4, and here $P(G, \lambda) = \lambda(\lambda - 1)^3 - \lambda(\lambda - 1)(\lambda - 2) = \lambda(\lambda - 1)(\lambda^2 - 3\lambda + 3)^{2-1}$. So the result follows for $n = 2$. Assuming the result true for an arbitrary (but fixed) $n \geq 1$, consider the situation for $n + 1$. Write $G = G_1 \cup G_2$, where G_1 is C_4 and G_2 is the ladder graph for n rungs. Then $G_1 \cap G_2 = K_2$, so from Theorem 11.14 we have $P(G, \lambda) = P(G_1, \lambda) \cdot \frac{P(G_2, \lambda)}{P(K_2, \lambda)} = \lambda(\lambda - 1)(\lambda^2 - 3\lambda + 3)^n$. Consequently, the result is true for all $n \geq 1$, by the Principle of Mathematical Induction.