HW3 Solutions

Ex 3.1: 2, 5, 10, 15, 29

Ex 3.2: 2, 4, 7, 17, 19

Ex 3.3: 4, 5, 6, 10

Ex 3.4: 4, 8, 9, 11, 15

Ex 3.1: (2)

* All of the statement are true except for part (f). Because the element 2 doesn’t exist in A.

Ex 3.1: (5)

a) {0, 2}
b) {2, 2(1/2), 3(1/3), 5(1/5), 7(1/7)}
c) {0, 2, 12, 36, 80}

Ex 3.1: (10)

* The nonempty sets are in parts (d), and (f).

Ex 3.1: (15)

* $W = \{1\} \in \{\{1\}, 2\} = X$
* $X = \{\{1\}, 2\} \in \{X, 3\} = Y$
* $W = \{1\} \notin \{X, 3\} = Y$

Ex 3.1: (29)

* **procedure** Subsets (i, j, k, l: positive integars)

```
begin
    for i := 1 to 4 do
        for j := i+1 to 5 do
            for k = j+1 to 6 do
                for l := k+1 to 7 do
                    print ({i, j, k, l})
end
```

Ex 3.2: (2)

a) $[2,3]$
b) $[0,7)$
c) $(-\infty,0) \cup (3, +\infty)$
d) $[0,2) \cup (3,7)$
e) $[0,2)$
f) $(3,7)$

Ex 3.2: (4)

a) True: (i), (iv), (v)
b) (i) E (ii) B (iii) D (iv) D
(v) $Z - A = \{2n+1 | n \in Z\}$ (vi) E

Ex 3.2: (7)

a) False. $Let\ \mathcal{U} = \{1,2,3\}, A = \{1\}, B = \{2\}, C = \{3\}.$
$Then\ A \cap B = B \cap C\ but\ A \neq B.$

b) False. $Let\ \mathcal{U} = \{1,2\}, A = \{1\}, B = \{2\}, C = \{1,2\}.$
$Then\ A \cup B = A \cup C\ but\ A \neq B.$

c) $x \in A \Longrightarrow x \in A \cup C \Longrightarrow x \in B \cup C. So\ x \in B\ or\ x \in C. If\ x \in B,$
$then\ we\ are\ finished. If\ x \in C, then\ A \cap C = B \cap C\ and\ x \in B.$
$In\ either\ case, x \in B\ so\ A \subseteq B.$
$Likewise, y \in B \Longrightarrow y \in B \cup C = A \cup C, so\ y \in A\ or\ y \in C.$
$If\ y \in C, then\ y \in B \cap C = A \cap C.$
$In\ either\ case, y \in A\ and\ B\ \subseteq A. Hence\ A = B.$

d) Let $x \in A$. Consider two cases:

1. $x \in C \Longrightarrow x \notin A \triangle C \Longrightarrow x \notin B \triangle C \Longrightarrow x \in B.$
2. $x \notin C \Longrightarrow x \in A \triangle C \Longrightarrow x \notin B \triangle C \Longrightarrow x \in B.$

In either case $A \subseteq B$. In a similar way we find $B \subseteq A$, so $A = B$

Ex 3.2: (17)

a) $A \cap (B - A) = A \cap \left(B \cap \overline{A}\right) = B \cap \left(A \cap \overline{A}\right) = B \cap \emptyset = \emptyset.$

b) $\left[(A \cap B) \cup \left(A \cap B \cap \overline{C} \cap D\right)\right] \cup \left(\overline{A} \cap B\right)$

$= (A \cap B) \cup \left(\overline{A} \cap B\right), \textit{by the Absorption}$

$= \left(A \cup \overline{A}\right) \cap B = \mathcal{U} \cap B = B$

c) $(A - B) \cup (A \cap B) = \left(A \cap \overline{B}\right) \cup (A \cap B) = A \cap \left(\overline{B} \cup B\right) = A \cap \mathcal{U} = A$

d) $\overline{A} \cup \overline{B} \cup \left(A \cap B \cap \overline{C}\right) = \overline{(A \cap B)} \cup \left[(A \cap B) \cap \overline{C}\right] =$

$\left[\overline{(A \cap B)} \cup (A \cap B)\right] \cap \left[\overline{(A \cap B)} \cup \overline{C}\right] = \left[\overline{(A \cap B)} \cup \overline{C}\right] = \overline{A} \cup \overline{B} \cup \overline{C}$

Ex 3.2: (19)

a) [-6, 9]
b) [-8, 12]
c) ∅
d) [-8, -6) ∪ (9, 12]
e) [-14, 21]
f) [-2, 3]
g) **R**
h) [-2, 3]

Ex 3.3: (4)

a) Here $A \cup B \cup C = C$, so $|A \cup B \cup C| = |C| = 5000.$

b) Here $A \cap B \cap C = \emptyset$ as well, so it follows from the formula for $|A \cup B \cup C| = |A| + |B| + |C| = 50 + 500 + 5000 = 5550.$

c) $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| = 5000 - 3 - 3 - 3 + 1 = 5542$

Ex 3.3: (5)

- $9! + 9! - 8!$

Ex 3.3: (6)

a) 12
b) 2
c) 16

Ex 3.3: (10)

* The number of arrangements with either H before E, or E before T, or T before M equals the total number of arrangements (i.e., 7!) minus the number of arrangements where E is before H, and T is before E and M is before T (i.e. **...M...T...E...H...**).
* There are 3! ways to arrange C, I, S. For each arrangement, there are four locations (i.e. **_C_I_S_**, locations as the underscores show) to select from, with repetition, to place M, T, E, H in this prescribed order. Hence, there are $(3!)\binom{4+4-1}{4} = (3!)\binom{7}{4}$ arrangements where M is before T, T before E, and E before H.
* Consequently, there are $7! - 3!\binom{7}{4}$ arrangements with either H before E, or E before T, or T before M.

Ex 3.4: (4)

- The probability of each equally likely outcome is $\frac{0.14}{7} = 0.02 = \frac{1}{n}$.
- Therefore, $n = \frac{1}{0.02} = 50$.

Ex 3.4: (8)

* $S = \{\{a, b, c\} | a, b, c \in \{1,2,3, \dots, 99,100\}, a \neq b, a \neq c, b \neq c\}$
* $A = \{\{a, b, c\} | \{a, b, c\} \in S, a + b + c \text{ } is \text{ } even\}$
  $= \{\{a, b, c\} | \{a, b, c\} \in S, a, b, c \text{ } are \text{ } even,$
  $or \text{ } one \text{ } of \text{ } a, b, c \text{ } is \text{ } even \text{ } and \text{ } the \text{ } other \text{ } two \text{ } are \text{ } odd\}$
* $|S| = \binom{100}{3} = 161{,}700$
* $|A| = \binom{50}{3} + \binom{50}{1}\binom{50}{2} = 80{,}850$
* Therefore, $\Pr(A) = \frac{1}{2}$

Ex 3.4: (9)

* The sample space $S = \{(x_1, x_2, \ldots, x_6) | x_i = H \text{ or } T, 1 \le i \le 6\}$. Hence $|S| = 2^6 = 64$.

a) The event $A = \{HHHHHH\}$ and $\Pr(A) = 1/64$.

b) The event $B = \{TTTTTH, TTTTHT, TTTHTT, TTHTTT, THTTTT, HTTTTT\}$ and $\Pr(B) = 3/32$.

c) There are $\frac{6!}{4!2!} = 15$ ways. The probability is $15/64$.

d) 0 head: 1 arrangement
2 heads: $\frac{6!}{4!2!} = 15$ arrangements
4 heads: $\frac{6!}{2!4!} = 15$ arrangements
6 heads: 1 arrangement
So the probability of the event is $(1 + 15 + 15 + 1)/64 = 1/2$

Ex 3.4: (9) (cont.)

- The sample space $S = \{(x_1, x_2, \ldots, x_6) |_{x_i} = H \; or \; T, 1 \leq i \leq 6\}$. Hence $|S| = 2^6 = 64$.

e) 4 heads: $\frac{6!}{4!2!} = 15$ arrangements
5 heads: $\frac{6!}{5!1!} = 6$ arrangements
6 heads: 1 arrangement
Hence, the probability is (15+6+1)/64 = 11/32.

Ex 3.4: (11.a)

* Let S $be\ the\ sample\ space$
  $S = \{(x_1, x_2, x_3) | 1 \leq x_i \leq 6, i = 1,2,3\}; so |S| = 6^3 = 216.$
* Let $A = \{(x_1, x_2, x_3) | x_1 < x_2\ and\ x_1 < x_3\}$
  $= \bigcup_{n=1}^{5} \{(n, x_2, x_3) | n < x_2\ and\ n < x_3\}.$
  For $1 < n < 5, |\{(n, x_2, x_3) | n < x_2\ and\ n < x_3\}| = (6 - n)^2.$
  Consequently, $|A| = 5^2 + 4^2 + 3^2 + 2^2 + 1^2 = 55.$
* Therefore, $\Pr(A) = 55/213$

Ex 3.4: (11.b)

* With S as in part (a), let $B = \{(x_1, x_2, x_3) | x_1 < x_2 < x_3\}$.
  Then $|\{(1, x_2, x_3) | 1 < x_2 < x_3\}| = 10,$
  $|\{(2, x_2, x_3) | 2 < x_2 < x_3\}| = 6,$
  $|\{(3, x_2, x_3) | 3 < x_2 < x_3\}| = 3,$
  $|\{(4, x_2, x_3) | 4 < x_2 < x_3\}| = 1,$
* Hence, $|B| = 20 \ and \ \Pr(B) = 20/216 = 5/54$

Ex 3.4: (15)

- $\Pr(A) = \frac{1}{3}, \Pr(B) = \frac{7}{15}, \Pr(A \cap B) = \frac{2}{15}, \Pr(A \cup B) = \frac{2}{3}.$
- $\Pr(A \cup B) = \frac{2}{3} = \frac{1}{3} + \frac{7}{15} - \frac{2}{15} = \Pr(A) + \Pr(B) - \Pr(A \cap B).$