Solution

Ex 4.1: 2, 8, 16, 19, 26

Ex 4.2: 1, 8, 10, 12, 16

Ex 4.3: 7, 15, 20, 22, 28

Ex 4.4: 1, 2, 7, 14, 19

Ex 4.5: 1, 2, 8, 24, 25

Ex 4.1: (2.a)

- $S(n): \sum_{i=1}^{n} 2^{i-1} = 2^n - 1$
- $S(1): \sum_{i=1}^{1} 2^{i-1} = 2^{1-1} = 2^1 - 1$, so $S(1)$ is true.
- Assume $S(k): \sum_{i=1}^{k} 2^{i-1} = 2^k - 1$ is true.
- Consider $S(k+1)$.

$\sum_{i=1}^{k+1} 2^{i-1} = \sum_{i=1}^{k} 2^{i-1} + 2^k = 2^k - 1 + 2^k = 2^{k+1} - 1$, so $S(k) \Longrightarrow S(k+1)$ and the result is true for all $n \in \mathbb{Z}^+$ by the Principle of Mathematical Induction.

Ex 4.1: (2.b)

- $S(n): \sum_{i=1}^{n} i2^i = 2 = 2 + (n-1)2^{n+1}$
- $S(1): \sum_{i=1}^{1} i2^i = 2 = 2 + (1-1)2^{1+1}$, so $S(1)$ is true.
- Assume $S(k): \sum_{i=1}^{k} i2^i = 2 + (k-1)2^{k+1}$ is true.
- Consider $S(k+1)$.

$\sum_{i=1}^{k+1} i2^i = \sum_{i=1}^{k} i2^i + (k+1)2^{k+1} = 2 + (k-1)2^{k+1} + (k+1)2^{k+1} = 2 + (2k)2^{k+1} = 2 + (k)2^{k+2}$,
so $S(k) \Longrightarrow S(k+1)$ and the result is true for all $n \in \mathbb{Z}^+$ by the Principle of Mathematical Induction.

Ex 4.1: (2.c)

- $S(n): \sum_{i=1}^{1}(i)(i!) = (n+1)! - 1$
- $S(1): \sum_{i=1}^{1}(i)(i!) = 1 = (1+1)! - 1$, so $S(1)$ is true.
- Assume $S(k): \sum_{i=1}^{k}(i)(i!) = (k+1)! - 1$ is true.
- Consider $S(k+1)$.

$\sum_{i=1}^{k+1}(i)(i!) = \sum_{i=1}^{k}(i)(i!) + (k+1)(k+1)! = (k+1)! - 1 + (k+1)(k+1)! = (k+2)! - 1$,
so $S(k) \Longrightarrow S(k+1)$ and the result is true for all $n \in \mathbb{Z}^{+}$
by the Principle of Mathematical Induction.

Ex 4.1: (8)

Here we have

$$\sum_{i=1}^{n} i^2 = \frac{(n)(n+1)(2n+1)}{6} = \frac{(2n)(2n+1)}{2} = \sum_{i=1}^{2n} i,$$

and $\frac{(n)(n+1)(2n+1)}{6} = \frac{(2n)(2n+1)}{2} \Longrightarrow n = 5.$

Ex 4.1(16.a & 16.b)

a) 3

b) $s_2 = 2;\ s_4 = 4$

Ex 4.1(16.c)

For $n \geq 1, sn = \sum_{\emptyset \neq A \subseteq Xn} \frac{1}{p_A} = n.$

Proof: For $n = 1, s_1 = \frac{1}{1} = 1$, so this first case is true and establishes the basis step. Now, for the inductive step, assume the result true for $n = k(\geq 1)$. That is, $s_{k+1} = \sum_{\emptyset \neq A \subseteq X_{k+1}} \frac{1}{p_A} = \sum_{\emptyset \neq B \subseteq X_k} \frac{1}{p_B} + \sum_{\{k+1\} \subseteq C \subseteq X_{k+1}} \frac{1}{p_C}$, where the first sum is taken over all nonempty subsets B of X_k and the second sum over all subsets C of X_{k+1} that contain $k+1$.

Then $s_{k+1} = s_k + \left[\frac{1}{k+1} + \frac{1}{k+1} s_k\right] = k + \frac{1}{k+1} + \frac{1}{k+1} k = k + 1$. Consequently, we have deduced the truth for $n = k + 1$ from that of $n = k$. The result follows for all $n >= 1$ by the Principle of Mathematical Induction.

Ex 4.1(19)

Assume $S(k)$ true for some $k \geq 1$.

For $S(k+1), \sum_{i=1}^{k+1} i = \frac{\left[k+\frac{1}{2}\right]^2}{2} + (k+1) = \frac{(k^2+k)+\frac{1}{4}+2k+2}{2} = \frac{\left[(k+1)^2+(k+1)+\frac{1}{4}\right]}{2} = \frac{\left[(k+1)+\frac{1}{2}\right]^2}{2}$. So $S(k) \Longrightarrow S(k+1)$.

However, we have no first value of k where $S(k)$ is true.

For each $k \geq 1, \sum_{i=1}^{k} i = \frac{(k)(k+1)}{2}$ and $\frac{(1)(1+1)}{2} = \frac{\left[1+\frac{1}{2}\right]^2}{2} \Longrightarrow 1 \neq \frac{9}{8}$.

Ex 4.1(26.a & 26.b)

a) $a_1 = \sum_{i=0}^{1-1} \binom{0}{i} a_i a_{(1-1)-i} = \binom{0}{0} a_0 a_0 = a_0{}^2$

$a_2 = \sum_{i=0}^{2-1} \binom{1}{i} a_i a_{(2-1)-i} = \binom{1}{0} a_0 a_1 + \binom{1}{1} a_1 a_0 = 2a_0{}^3.$

b) $a_3 = \sum_{i=0}^{3-1} \binom{3-1}{i} a_i a_{(3-1)-i} = \sum_{i=0}^{2} \binom{2}{i} a_i a_{2-i} =$
$\binom{2}{0} a_0 a_2 + \binom{2}{1} a_1 a_1 + \binom{2}{2} a_2 a_0 = (a_0)(2a_0{}^3) +$
$2(a_0{}^2)(a_0{}^2) + (2a_0{}^3)(a_0) = 6a_0{}^4$

$a_4 = \sum_{i=0}^{4-1} \binom{4-1}{i} a_i a_{(4-1)-i} = \sum_{i=0}^{3} \binom{3}{i} a_i a_{3-i} =$
$\binom{3}{0} a_0 a_3 + \binom{3}{1} a_1 a_2 + \binom{3}{2} a_2 a_1 + \binom{3}{3} a_3 a_0 =$
$(a_0)(6a_0{}^4) + 3(a_0{}^2)(2a_0{}^3) + 3(2a_0{}^3)(a_0{}^2) +$
$(6a_0{}^4)(a_0) = 24a_0{}^5$

Ex 4.1(26.c)

For $n \geq 0, a_n = (n!)a_0^{n+1}$.

Proof: (By the Alternative Form of the Principle of Mathematical Induction) The result is true for $n = 0$ and this establishes the basis step. [In fact, the calculations in parts (a) and (b) show the result is also true for $n = 1,2,3$ and 4.] Assuming the result true for $n = 0,1,2,3, \ldots, k(\geq 0)$ – that is, that $a_n = (n!)a_0^{n+1}$ for $n = 0,1,2,3, \ldots, k(\geq 0)$ – we find that

$$a_{k+1} = \sum_{i=0}^{k} \binom{k}{i} a_i a_{k-i} = \sum_{i=0}^{k} \binom{k}{i} (i!)\left(a_0^{i+1}\right)(k-i)!\left(a_0^{k-i+1}\right) = \sum_{i=0}^{k} \binom{k}{i} (i!)(k-i)!\, a_0^{k+2} = \sum_{i=0}^{k} k!\, a_0^{k+2} = (k+1)\left[k!\, a_0^{k+2}\right] = (k+1)!\, a_0^{k+2}.$$

So the truth of the result for $n = 0,1,2, \ldots, k(\geq 0)$ implies the truth of the result for $n = k + 1$. Consequently, for all $n \geq 0, a_n = (n!)a_0^{n+1}$ by the Alternative Form of the Principle of Mathematical Induction.

Ex 4.2(1)

a) $c_1 = 7;\ and\ c_{n+1} = c_n + 7, for\ n \geq 1.$
b) $c_1 = 7;\ and\ c_{n+1} = 7c_n, for\ n \geq 1.$
c) $c_1 = 10;\ and\ c_{n+1} = c_n + 3, for\ n \geq 1.$
d) $c_1 = 7;\ and\ c_{n+1} = c_n, for\ n \geq 1.$
e) $c_1 = 1;\ and\ c_{n+1} = c_n + 2n + 1, for\ n \geq 1.$
f) $c_1 = 3, c_2 = 1;\ and\ c_{n+2} = c_n, for\ n \geq 1.$

Ex 4.2(8.a)

1) For $n = 2, x_1 + x_2$ denotes the ordinary sum of the real numbers x_1 and x_2.
2) For real number $x_1, x_2, \ldots, x_n, x_{n+1}$, we have
$x_1 + x_2 + \cdots + x_n + x_{n+1} = (x_1 + x_2 + \cdots + x_n) + x_{n+1}$,
the sum of the two real number $x_1 + x_2 + \cdots + x_n$ and x_{n+1}

Ex 4.2(8.b)

The truth of this result for $n = 3$ follows from the Associative Law of Addition – since $x_1 + (x_2 + x_3) = (x_1 + x_2) + x_3$, there is no ambiguity in writing $x_1 + x_2 + x_3$. Assuming the result true for all $k \geq 3$ and all $1 \leq r < k$, let us examine the case for $k + 1$ real numbers. We find that

1) $r = k$ we have $(x_1 + x_2 + \cdots + x_r) + x_{r+1} = x_1 + x_2 + \cdots + x_r + x_{r+1}$ by virtue of the recursive definition.
2) For $1 \leq r < k$ we have

$$\begin{aligned}(x_1 + x_2 + \cdots + x_r) + (x_{r+1} + \cdots + x_k + x_{k+1}) \\ &= (x_1 + x_2 + \cdots + x_r) + [(x_{r+1} + \cdots + x_k) + x_{k+1}] \\ &= [(x_1 + x_2 + \cdots + x_r) + (x_{r+1} + \cdots + x_k)] + x_{k+1} \\ &= (x_1 + x_2 + \cdots + x_r + x_{r+1} + \cdots + x_k) + x_{k+1} \\ &= x_1 + x_2 + \cdots + x_r + x_{r+1} + \cdots + x_k + x_{k+1}.\end{aligned}$$

So the result is true for all $n \geq 3$ and all $1 \leq r < n$, by the Principle of Mathematical Induction.

Ex 4.2(10)

The result is true for $n = 2$ by the material presented at the start of the problem. Assuming the truth for $n = k$ real numbers, we have, for
$n = k, |x_1 + x_2 + \cdots + x_k + x_{x+1}| =$
$|(x_1 + x_2 + \cdots + x_k) + x_{x+1}| \leq$
$|x_1 + x_2 + \cdots + x_k| + |x_{x+1}| \leq$
$|x_1| + |x_2| + \cdots + |x_k| + |x_{x+1}|,$
so the result is true for all $n \geq 2$ by the Principle of Mathematical Induction.

Ex 4.2(12)

Proof: (By Mathematical Induction)

We find that $F_0 = \sum_{i=0}^{0} F_i = 0 = 1 - 1 = F_2 - 1$, so the given statement holds in this first case $-$ and this provides the basis step of the proof.

For the induction step we assume the truth of the statement when $n = k(\geq 0) -$ that is , that $\sum_{i=0}^{k} F_i = F_{k+2} - 1$.

Now we consider what happens when $n = k + 1$. We find for this case that $\sum_{i=0}^{k+1} F_i = \left(\sum_{i=0}^{k} F_i\right) + F_{k+1} = (F_{k+2} + F_{k+1}) - 1 = F_{k+3} - 1$, so the truth of the statement at $n = k$ implies the truth at $n = k + 1$.

Consequently, $\sum_{i=0}^{n} F_i = F_{n+2} - 1$ for all $n \in \mathrm{N} -$ by the Principle of Mathematical Induction.

Ex 4.2(16)

a) Let E denote the set of all positive even integers. We define E recursively by
   1) $2 \in \text{E}$; and
   2) For each $n \in \text{E}$, $n + 2 \in \text{E}$.

b) If G denotes the set of all nonnegative even integers. We define G recursively by
   1) $0 \in \text{G}$; and
   2) For each $m \in \text{G}$, $m + 2 \in \text{G}$.

Ex 4.3(7)

a) $(a, b, c) = (1,5,2)\ or\ = (5,5,3)$ ...

b) Proof:

$31|(5a + 7b + 11c) \Longrightarrow 31|(10a + 14b + 22c)$.

Also, $31|(31a + 31b + 31c)$,

so $31|[(31a + 31b + 31c) - (10a + 14b + 22c)]$.

Hence $31|(21a + 17b + 9c)$.

Ex 4.3(15)

	Base 10	Base 2	Base 16
(a)	22	10110	16
(b)	527	1000001111	20F
(c)	1234	10011010010	4D2
(d)	6923	1101100001011	1B0B

Ex 4.3(20)

a) 00001111
b) 11110001
c) 01100100
d) At Right
e) 01111111
f) 10000000

(d)	
Start with the binary representation of 65	65 ↓ 01000001
Interchanges the 0’s and 1’s to obtain the one’s complement	↓ 10111110
Add 1 to the one’s complement	↓ 10111111

Ex 4.3(22)

(a)
$$\begin{array}{r l}
0101 & = 5 \\
\underline{+0001} & = 1 \\
0110 & = 6
\end{array}$$

(b)
$$\begin{array}{r l}
1101 & = -3 \\
\underline{+1110} & = -2 \\
1011 & = -5
\end{array}$$

(c)
$$\begin{array}{r l}
0111 & = 7 \\
\underline{+1000} & = -8 \\
1111 & = -1
\end{array}$$

(d)
$$\begin{array}{r l}
1101 & = -3 \\
\underline{+1010} & = -6 \\
0111 & \neq -9 \quad \text{overflow error}
\end{array}$$

Ex 4.3(28)$_{1/2}$

Proof: Let $Y = \{3k | k \in \mathrm{Z}^+\}$, the set of all positive integers divisible by 3. In order to show that $X = Y$ we shall verify that $X \subseteq Y$ and $Y \subseteq X$.

(i) ($X \subseteq Y$): By part (1) of the recursive definition of X we have 3 in X. And since $3 = 3 \cdot 1$, it follows that 3 is in Y. Turning to part (2) of this recursive definition suppose that for $x, y \in \mathrm{X}$ we also have $x, y \in \mathrm{Y}$. Now $x + y \in \mathrm{X}$ by the definition and we need to show that $x + y \in \mathrm{Y}$. This follows because $x, y \in \mathrm{Y} \Rightarrow x = 3m, y = 3n$ for some $m, n \in \mathrm{Z}^+ \Rightarrow x + y = 3m + 3n = 3(m + n)$, with $m + n \in \mathrm{Z}^+ \Rightarrow x + y \in \mathrm{Y}$. Therefore every positive integer that results from either part (1) or part (2) of the recursive definition of X is an element in Y, and, consequently, $X \subseteq Y$.

Ex 4.3(28)$_{2/2}$

(ii) ($Y \subseteq X$): In order to establish this inclusion we need to show that every positiveinteger multiple of 3 is in X. This will be accomplished by the Principle of Mathematical Induction.

Start with the open statement

$$S(n): 3n \text{ is an element in } X,$$

which is defined for the universe Z^+. The basis step $-$ that is, $S(1)$ $-$ is true because $3 \cdot 1 \;=\; 3$ is in X by part (1) of the recursive definition of X.

For the inductive step of this proof we assume the truth of $S(k)$ for some $k(\geq 1)$ and consider what happens at $n = k + 1$. From the inductive hypothesis $S(k)$ we know that $3k$ is in X. Then from part (2) of the recursive definition of X we find that $3(k+1) = 3k + 3 \in \mathrm{X}$ because $3k, 3 \in \mathrm{X}$. Hence $S(k) \Rightarrow S(k+1)$.

So by the Principle of Mathematical Induction it follows that $S(n)$ is true for all $n \in \mathrm{Z}^+$ $-$and, consequently, $Y \subseteq X$. With $X \subseteq Y$ and $Y \subseteq X$ it follows that $X = Y$.

Ex 4.4(1)

a) 1820 = 7(231) + 203
   231 = 1(203) + 28
   203 = 7(28) + 7
   28 = 7(4), so gcd(1820,23) = 7
   1 = 203 - 7(28) = 203 -7[231- 203] = (-7)(231) + 8(203)
   = (-7)(231) + 8[1820 -7(231)] = 8(1820) + (-63)(231)

b) gcd(1369,2597) = 1 = 2597(534) + 1369(-1013)

c) gcd(2689,4001) = 1 = 4001(-1117) + 2689(1662)

Ex 4.4(2)

a) If as+bt= 2, then gcd(a,b) = 1 or 2, for the gcd of a,b divides a,b so it divides as+bt=2.
b) as+bt=3 $\Rightarrow$ gcd(a,b)=1 or 3.
c) as+bt=4 $\Rightarrow$ gcd(a,b)=1,2 or 4.
d) as+bt=6 $\Rightarrow$ gcd(a,b)=1,2,3 or 6.

Ex 4.4(7)

- Let $\gcd(a, b) = h, \gcd(b, d) = g$.
  $\gcd(a, b) = h$
  $\Rightarrow h|a, h|b$
  $\Rightarrow h|(a \cdot 1 + bc) \Rightarrow h|d$.
- $h|b, h|d$
  $\Rightarrow h|g. \gcd(b, d) = g$
  $\Rightarrow g|b, g|d$
  $\Rightarrow g|(d \cdot 1 + b(-c))$
  $\Rightarrow g|a. g|b, g|a, h = \gcd(a, b)$
  $\Rightarrow g|h. h|g, g|h, with\ g, h \in \mathrm{Z}^{+}$
  $\Rightarrow g = h$

Ex 4.4(14)

* $33x + 29y = 2490$
  $\gcd(33,29) = 1, and\ 33 = 1(29) + 4, 29 = 7(4) + 1, so\ 1$
  $= 29 - 7(4) = 29 - 7(33 - 29) = 8(29) - 7(33). 1$
  $= 33(-7) + 29(8) \Rightarrow 2490 = 33(-17430) + 29(19920)$
  $= 33(-17430 + 26k) + 29(19920 - 33k), for\ all\ k \in \mathrm{Z}.$
* $x = -17430 + 29k, y = 19920 - 33k$
  $x \geq 0 \Rightarrow 29k \geq 17430 \Rightarrow k \geq 602$
  $y \geq 0 \Rightarrow 19920 \geq 33k \Rightarrow 603 \geq \mathrm{k}$
* $k = 602: x = 28, y = 54;$
  $k = 603: x = 57, y = 21$

Ex 4.4(19)

* From Theorem 4.10 we know that $ab = lcm(a,b) \cdot \gcd(a,b)$.
* Consequently,

$$b = \frac{[lcm(a,b) \cdot \gcd(a,b)]}{a} = \frac{(242{,}500)(105)}{630}$$

Ex 4.5(1)

a) $2^2 \cdot 3^3 \cdot 5^3 \cdot 11$

b) $2^4 \cdot 3 \cdot 5^2 \cdot 7^2 \cdot 11^2$

c) $3^2 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$

Ex 4.5(2)

$\gcd(148500,7114800) = 2^2 3^1 5^2 11^1 = 3300$
$lcm(148500,7114800) = 2^4 3^3 5^3 7^2 11^2 = 320166000$
$\gcd(148500,7882875) = 3^2 5^3 11^1 = 12375$
$lcm(148500,7882875) = 2^2 3^3 5^3 7^2 11^1 13^1$
$= 94594500$
$\gcd(7114800,7882875) = 3^1 5^2 7^2 11^1 = 40425$
$lcm(7114800,7882875) = 2^4 3^2 5^3 7^2 11^2 13^1$
$= 1387386000$

Ex 4.5(8)

a) There are (15)(10)(9)(11)(4)(6)(11)=3920400 positive divisors of $n = 2^{14}3^{9}5^{8}7^{10}11^{3}13^{5}37^{10}$.

b) (i) (14-3+1)(9-4+1)(8-7+1)(10-0+1)(3-2+1)(5-0+1)(10-2+1)=(12)(6)(2)(11)(2)(6)(9)=171072
(ii) Since 1166400000=$2^9 3^6 5^5$, the number of divisors here is (14-9+1)(9-6+1)(8-5+1)(10-0+1)(3-0+1)(5-0+1)(10-0+1)=(6)(4)(4)(11)(4)(6)(11)=278784
(iii)(8)(5)(5)(6)(2)(3)(6)=43200
(iv)(7)(3)(4)(6)(1)(3)(6)=9072
(v) (5)(4)(3)(4)(2)(2)(4)=3840
(vi)(1)(1)(2)(2)(1)(1)(3)=12
(vii)(3)(2)(2)(2)(1)(1)(2)=48

Ex 4.5(24)

a) $\prod_{i=1}^{5}(i^2 + i)$
b) $\prod_{i=1}^{5}(1 + x^i)$
c) $\prod_{i=1}^{6}(1 + x^{2i-1})$

Ex 4.5(25)

Proof: (By mathematical Induction)

For n $= 2$ we find that $\prod_{i=2}^{2}(1-\frac{1}{i^2}) = \left(1-\frac{1}{2^2}\right) = \left(1-\frac{1}{4}\right) = \frac{3}{4} = \frac{2+1}{2\cdot 2}$, so the result is true in this first case and this establishes the basis step for our inductive proof.

Next we assume the result true for some (particular) k $\in$ Z^+ where k ≥ 2.

This gives us $\prod_{i=2}^{k}(1-\frac{1}{i^2}) = \frac{k+1}{2k}$. When we consider the case for n $=$ k $+ 1$, using the inductive step, we find that

$\prod_{i=2}^{k+1}\left(1-\frac{1}{i^2}\right) = \left(\prod_{i=2}^{k}\left(1-\frac{1}{i^2}\right)\right)\left(1-\frac{1}{(k+1)^2}\right) = \frac{\mathrm{k}+2}{2(\mathrm{k}+1)} = \frac{(k+1)+1}{2(k+1)}$.

The result now follows for all positive integers n ≥ 2 by the Principle of Mathematical Induction.