Name: Student ID:

Quiz #4 (5% + 2% Bonus Point)

CS2336 Discrete Mathematics, Instructor: Cheng-Hsin Hsu

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Take-home, Please Turn in by 12:00 p.m. (noon) on April 2nd, 2014

1) (1%) Give a recursive definition of the set of all
   a) negative even integers.
   b) perfect squares of all integers.

   Solution:

   (a) Let A denote the set of all negative even integers.

   $-2 \in A$; and $\forall a \in A, a - 2 \in A$

   (b) Let B denote the set of all perfect squares of all integers.

   $0 \in B$*; and* $\forall b \in B, b + 2\sqrt{b} + 1 \in B$

2) *(2%)*
   a) *(1%) How many positive divisors are there for* $n = 2^{14}3^9 5^8 7^{10} 11^3 13^5 37^{10}$*?*
   b) *(0.5%) How many of them are perfect cubes?*
   c) *(0.5%) How many of them are divisible of 1,166,400,000?*

   Solution:

   a) $15 \times 10 \times 9 \times 11 \times 4 \times 6 \times 11 = 3920400$

   b) $5 \times 4 \times 3 \times 4 \times 2 \times 2 \times 4 = 3840$

c) $6 \times 4 \times 4 \times 11 \times 4 \times 6 \times 11 = 278784$

3) (2%) Prove the following equations for $n \geq 1$ using mathematical induction.

a) $\sum_{i=1}^{n}(i)(i!) = (n+1)! - 1$.

b) $\sum_{i=1}^{n}(2^i)i - 2 = (n-1)2^{n+1}$.

Solution:

a) *(1)* $n = 1$, $\sum_{i=1}^{1}(i)(i!) = (n+1)! - 1$ *is true*

(2) Assume $n = k$ *is true:* $\sum_{i=1}^{k}(i)(i!) = (k+1)! - 1$

Consider $n = k+1$*:*

$\sum_{i=1}^{k+1}(i)(i!) = \sum_{i=1}^{k+1}(i)(i!) + (k)(k+1)! = (k+1)! - 1 + (k+1)(k+1)! = (k+2)! - 1$

From (1)(2), $\sum_{i=1}^{n}(i)(i!) = (n+1)! - 1$ *is true by the Principle of Mathematical Induction.*

b) *(1)* $n = 1$, $\sum_{i=1}^{1}(2^i)i - 2 = (1-1)2^{1+1}$ *is true*

(2) Assume $n = k$ *is true:* $\sum_{i=1}^{k}(2^i)i - 2 = (k-1)2^{k+1}$

Consider $n = k+1$*:*

$\sum_{i=1}^{k+1}(2^i)i - 2 = \sum_{i=1}^{k}(2^i)i - 2 + 2^{k+1}(k+1) = (k-1)2^{k+1} + 2^{k+1}(k+1)$

$[(k+1) - 1]2^{(k+1)+1}$

From (1)(2), $\sum_{i=1}^{k+1}(2^i)i - 2 = (n-1)2^{n+1}$ *is true by the Principle of Mathematical Induction.*

4) *(2%) Define the set* $X \subseteq \mathbb{Z}^+$ *as follows: (i)* $5 \in X$ *and (ii) if* $a, b \in X$*, then* $a + b \in X$*. Prove that* X *is the set of all positive integers divisible by* 5*.*

Solution:

Let $Y = 5k | k \in \mathbb{Z}^+$, the set of all positive integers divisible by 5. In order to show that $X = Y$ we shall verify that $X \subseteq Y and Y \subseteq X$.

a) ($X \subseteq Y$) By part (1) of the recursive definition of X we have 3 in X. And since $5 = 5 \times 1$, it follows that 5 is in Y . Turning to part (2) of this recursive definition suppose that for $x, y \in X$ we also have $x, y \in Y$. Now $x+y \in X$ by the definition and we need to show that $x + y \in Y$. This follows because $x, y \in Y \Rightarrow x = 5m, y = 5n$ for some $m, n \in \mathbb{Z}^+ \Rightarrow x+y = 5m+5n = 5(m+n)$, with $m+n \in \mathbb{Z}^+ \Rightarrow x+y \in Y$.

Therefore every positive integer that results from either part (1) or part (2) of the recursive definition of X is an element in Y , and, consequently, $X \subseteq Y$.

b) ($Y \subseteq X$) In order to establish this inclusion we need to show that every positive integer multiple of 3 is in X. This will be accomplished by the Principle of Mathematical Induction. Start with the open statement $S(n)$: $3n$ is an element in X, which is defined for the universe $\mathbb{Z}^{+}$. The basis step - that is, $S(1)$ - is true because $5 \times 1 = 5$ is in X by part (1) of the recursive definition of X. For the inductive step of this proof we assume the truth of $S(k)$ for some $k \geq 1$ and consider what happens at $n = k + 1$. From the inductive hypothesis S(k) we know that 5k is in X. Then from part (2) of the recursive definition of X we fond that $5(k + 1) = 5k + 5 \in X$, because $5k, 5 \in X$. Hence $S(k) \Rightarrow S(k + 1)$. So by the Principle of Mathematical Induction it follows that $S(n)$ is true for all $n \in \mathbb{Z}^{+}$- and, consequently, $Y \subseteq X$.

With $X \subseteq Y$ and $Y \subseteq X$ it follows that $X = Y$.