Department of Computer Science
National Tsing Hua University

CS 2336: Discrete Mathematics

Instructor: Cheng-Hsin Hsu

Course Information

- **TextBook**: Discrete and Combinatorial Mathematics, R. Grimaldi, 5th Ed., Addison Wesley
- **Lecture**: Mondays 10:10 a.m.-12:00 p.m. and Thursdays 9:00-9:50 a.m.
- **Location**: 台達館 106
- **Office Hour:** Wednesdays 10:00 a.m. – 12:00 p.m.
- **Course Website**:

  http://nmsl.cs.nthu.edu.tw/index.php/courses
- **TA**: Shu-Ting Wang (kelvinstwang@gmail.com)
- **TA Office Hour:** Mondays 12:00 - 1:00 p.m. at Delta 106

Topics We Plan to Cover

1. Fundamental Principles of Counting (Chapter 1 in textbook)
2. Fundamental of Logic (Chapter 2)
3. Set Theory (Chapter 3)
4. Properties of Integers: Mathematical Induction (Chapter 4)
5. Relations and Functions (Chapter 5)
6. Languages: Finite State Machines (Chapter 6)
7. Relations: The Second Time Around (Chapter 7)
8. The Principle of Inclusion and Exclusion (Chapter 8)
9. Generating Functions (Chapter 9)
10. Recurrence Relations (Chapter 10)
11. An Introduction to Graph Theory (Chapter 11)
12. Trees (Chapter 12)
13. Optimization and Matching (Chapter 13)

Fundamentals (1–7)
Enumeration (8–10)
Graph Theory (11–13)

Not Covered: Applied Algebra

Grading Policy

- **Quizzes (60%):** One for each chapter
  - Given on Mondays, either at the beginning or end of the lecture
  - Sample questions will be given as homework, which are not collected nor graded
  - No makeup quizzes, unless an email requesting for a leave is sent to and approved by the instructor before each quiz
- **Midterm Exam (20%)**: 2-hr exam on Nov. 3rd
- **Final Exam (20%)**: 2-hr exam on Jan. 12th

Tentative Schedule

Week	Mondays	Wednesdays	Homework/Quiz Solutions
1: Sep 14	Introduction PDF	Ch. 1 Fundamental Principles of Counting PDF	
2: Sep 21	Ch. 1 Fundamental Principles of Counting PDF	Ch. 2 Fundamental of Logic PDF	
3: Sep 28	Ch. 2 Fundamental of Logic PDF	Ch. 2 Fundamental of Logic PDF	
4: Oct 5	Ch. 2 Fundamental of Logic PDF	Ch. 3 Set theory PDF	
5: Oct 12	Ch. 4 Properties of Integers: Mathematical Induction PDF	Conference Travel	
6: Oct 19	Ch. 4 Properties of Integers: Mathematical Induction PDF	Ch. 5 Relations and Functions PDF	
7: Oct 26	Ch. 5 Relations and Functions PDF	Ch. 5 Relations and Functions PDF	
8: Nov 2	Ch. 5 Relations and Functions PDF	Ch. 6 Finite State Machines PDF	
9: Nov 9	Mid Term Exam (Chs. 1 - 5)	Conference Travel (Mid Term Review)	
10: Nov 16	Ch. 7 Relations: The Second Time Around PDF	Ch. 7 Relations: The Second Time Around PDF	
11: Nov 23	Ch. 7 Relations: The Second Time Around PDF	Ch. 7 Relations: The Second Time Around PDF	
12: Nov 30	Ch. 8 The Principle of Inclusion and Exclusion PDF	Ch. 8 The Principle of Inclusion and Exclusion PDF	
13: Dec 7	Ch. 8 The Principle of Inclusion and Exclusion PDF	Ch. 8 The Principle of Inclusion and Exclusion PDF	
14: Dec 14	Ch. 9 Generating Functions PDF	Ch. 9 Generating Functions PDF	
15: Dec 21	Ch. 9 Generating Functions PDF	Ch. 10 Recurrence Relations PDF	
16: Dec 28	Holiday	Ch. 10 Recurrence Relations PDF	
17: Jan 4	Ch. 11 An Introduction to Graph Theory PDF	Ch. 11 An Introduction to Graph Theory PDF	
18: Jan 11	Final Exam (Chs. 6 - 11)		

What is Discrete Mathematics?

- Covers various kinds of topics
  - Logics
  - Combinatorial
  - Algorithms
  - Graph Theory
  - Number Theory
- Discrete $\leftarrow$ something you can count

What is a Proof?

- Vaguely speaking:
  - To convince someone that something is true
- Mathematical proof:
  - To show if some axioms are true, then some statements are also true
- What we will do in this course is somewhere between these two definitions
  - You need to get the main idea
  - Details are not important

Examples of Proofs (1/2)

- Prove that $\sqrt{2}$ is irrational (by contradiction)
  - Assume $\sqrt{2}$ is a rational number, then we write $\sqrt{2}$ $=p/q$, where p and q are the lowest terms
  - We know $2 = p^2/q^2$ and $2q^2 = p^2$ → p is even
  - We write $p=2k$, then we have $2q^2 = 4k^2$ and $q^2 = 2k^2$
  - Then, q is also even → contradiction!
  - Hence we know $\sqrt{2}$ is irrational

Examples of Proofs (2/2)

- Prove that there are infinite prime numbers [Eulclid 300 BC]
  - A prime is a nature number with exactly two divisors
  - Assume there are finite # of primes: p_1, p_2, …., $p_{n,}$ sorted in the increasing order
  - We create a new prime $p = p_1 p_2 \cdots p_n + 1$
    - Why p is a prime? → p / p_i for any *i* has remainder *1*
  - Moreover, p is larger then p_1, p_2, …., $p_{n,}$ so p is a new prime → contracting to the assumption that there are only n prime numbers out there
  - Hence, there are infinite number of prime numbers

How Math Abstraction Helps?

- Consider two problems
  - Pick 2 out of 6 students, how many different ways can we choose?
  - How many bowling pins do we have if there are 5 rows?
- Connect them using triangular number
  - $T_n = 1 + 2 + \cdots + n$
- Prove $T_n = \dfrac{n(n+1)}{2}$
  - Geometrical
  - Algebraic

Questions?

Questions so far?

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Chapter 1

Fundamental Principles of Counting

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Rule of Sum

- Q: There are 6, 8, and 12 introductory books for C ++, Java, and Perl, respectively. If Joe wants to learn a first language, how many choices (of books) does he have?
- Q: Say Joe has two friends, who own 3 and 5 of these books. How many unique books can Joe borrow from his friends?
- Rule: If task 1 can be performed in m ways, task 2 can be performed in n ways, and tasks 1 and cannot be performed concurrently, then there are $m+n$ ways to perform either task 1 or 2.

Rule of Product

- Q: There is a singer try-out with 4 men and 6 women, how many different way to form a couple of singers?
- Q: How many different license plates of two letters followed by four digits can we produce with/without repetition
- Rule: If task 1 can be performed in *m* ways, task 2 can be performed in *n* ways, there are mn ways to sequentially perform tasks 1 and 2.

Combinations of Two Rules

- Q: A cafeteria offers 6 kinds of muffins, 8 kinds of sandwiches, and 3 different beverages, and each combo consists of a muffin (or sandwiches) and a beverage. How many different combos does this cafeteria offer?

Permutation

- Q: From a class of 10 students, choose and seat 5 of them in a line for a picture. How many linear arrangement are possible?
  - Write the solution using *n* factorial
- Definition: Given a collection of *n* distinct object, any arrangement of these objects is called a permutation of the collection
- For *n* distinct objects and an integer $1 \leqslant r \leqslant n$, per the rule of product, the number of permutation of size *r* of *n* object is: $P(n, r) = \frac{n!}{(n-r)!}$

Repeated Permutation

- Q: How many 5-character permutations of the letters in "COMPUTER" if
  - Repetitions are not allowed
  - Repetitions are allowed
- Q: How many permutations of the letters in
  - BALL
  - DATABASES
- General Result: If there are n objects with n_1 (indistinguishable) type 1 objects,…, n_r type r objects, there are $\frac{n!}{n_1!n_2!\cdots n_r!}$ arrangements of the given n objects

Nonlinear Permutation

- Q: If six students are seated at a round table, how many different arrangements are possible?
  - Assume that arrangements are considered the same when once can be derived by rotating the other

- Q: How many ways we can arrange 3 males and 3 female around a table so that the sexes alternate?

Combinations

- Q: How many possible permutations of 3 from 5 cards? What if the cards are considered as unordered?
- General rule: For *n* distinct object, the number of combinations of *r* object, where $0 \leqslant r \leqslant n$, is:

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$$

- Q: How many different ways a student can answer 7 out of 10 questions in a test?
- Q: What if the student needs to pick 3 questions from the first 5 questions and 4 questions from the last 5 questions?

Permutation and/or Combination

- Q: How many arrangements of the letters in TALLAHASSEE with no adjacent A's?
  - Requires both permutation and combination
- Q: The coach wants to form four teams of 9 students each from a class of 36 students. Call the teams A, B, C, and D, how many different ways can the coach form the teams?
  - Solve it using combinations
  - Solve it using permutations

Summation

- We use Greek symbol sigma to represent summations
  - $\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_n$
  - i is the index
  - m is the lower limit
  - n is the upper limit
- Example:

$$\sum_{i=2}^{3} \binom{5}{7-j}\binom{5}{j} = \binom{5}{5}\binom{5}{2} + \binom{5}{4}\binom{5}{3}$$

Another Example

- Q: Let n be a positive integer, there are 3^n strings of an alphabet consisting of symbols 0, 1, and 2. Define weight(x)=$x_1+x_2+\ldots+x_n$, for $x=x_1x_2\ldots x_n$. For $n=10$, how many strings with even weights?

Theorems

- Lemma: Prove that for two integers *n* and *r*, $n \geqslant r \geqslant 0$, we have $\binom{n}{r} = \binom{n}{n-r}$

- Theorem (Binomial): Prove that

$$(x+y)^n = \binom{n}{0}x^0y^n + \binom{n}{1}x^1y^{n-1} + \cdots + \binom{n}{n}x^ny^0 = \sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}$$

  - $\binom{n}{k}$ is called binomial coefficient

- Corollary
  - $2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$
  - $0 = \binom{n}{0} - \binom{n}{1} + \cdots + (-1)^n\binom{n}{n}$

Theorems (cont.)

- Theorem (Multinomial): Prove that the coefficient of $x_1^{n_1} x_2^{n_2} \dots x_t^{n_t}$ in the expansion of $(x_1+x_2+\dots+x_t)^n$ is $\dfrac{n!}{n_1!n_2!\dots n_t!}$

- $\binom{n}{n_1, n_2, \dots, n_t}$ is called multinomial coefficient

- Example: Determine the coefficient of $a^2b^3c^2d^5$ in the expansion of $(a + 2b - 3c + 2d + 5)^{16}$

Comb. with Repetition

- Q: Seven students stop at a fast food restaurant where each of them can order a burger, a hot dog, a taco, or a sandwich. The restaurant only cares about how many burgers, hot dogs, tacos, and sandwiches do the students order. What is the number of possible solutions?
- General Rule: The number of combinations of, with repetition, r objects from n distinct objects is:

$$\frac{(n+r-1)!}{r!(n-1)!} = \binom{n+r-1}{r}$$

Examples

- Q: A donut shop offer 20 kinds of donuts. Assuming that there are plenty of donuts of each kind, how many ways for a kid to buy a dozen donuts?
- Q: How many solutions does $x_1 + x_2 + x_3 + x_4 = 7$ have, for positive $x_{1,}$ x_2, x_3, *and* x_4?
- Q: A father distributes $1000 among 4 kids at a step of $100
  - How many different ways to distribute $1000 if some kids may get nothing?
  - How many ways to distribute $1000 if each kid is guaranteed to have at least $100?

Equivalence

- The number of selections, with repetition, of size *r* from a collection of size *n*
- The number of ways *r* identical objects can be distributed among *n* distinct containers
- The number of integer solutions of the equation

$$x_1 + x_2 + \cdots + x_n = r, \quad x_i \geqslant 0, 1 \leqslant i \leqslant n$$

Examples

- Q: How many nonnegative solutions does the inequality $x_1 + x_2 + \cdots + x_6 < 10$ have?

- Q: How many times the print statement is executed?

```
for i = 1 to 20
    for j = 1 to i
        for k = 1 to j
            print (i+j+k);
```

Take-home Exercise

- Exercise 1.1 and 1.2: 15, 22, 28, 32, 33
- Exercise 1.3: 13, 16, 25, 29, 34
- Exercise 1.4: 7, 17, 24, 26, 28