Department of Computer Science
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# CS 2336: Discrete Mathematics

## Chapter 10

## Recurrence Relations

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# Outline

10.1 The First-Order Linear Recurrence Relation

10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

10.3 The Nonhomogeneous Recurrence Relation

10.4 The Method of Generating Functions

# Geometric Progression

- For a sequence, we want to write $a_n$ as a function of the prior terms $a_0, a_1, \ldots, a_{n-1}$
- Geometric progression: an infinite sequence with a common ratio
  - For example: 5, 15, 45, 135, …, where $a_{n+1}=3a_n$, and $a_0=5$
- $a_{n+1}=3a_n$ is the recurring relation, 3 is the common ratio, and $a_0$ helps us to determine the right sequence
  - Many sequences can be generated with a recurring realtion

# Terminology

- A recurrence relation is first order linear homogeneous with constant coefficients, if $a_{n+1}$ (current term) only depends on $a_n$ (previous term)
- A known term $a_0$ or $a_1$, is called the boundary condition
  - If $a_0$ equals to a constant, it is also called initial condition
- Example, $a_{n+1}=3a_n$, $a_0=5$
  - Unique solution: $a_n=5(3^n)$
  - No longer need to compute $a_5$ before getting $a_6$

# General Form and Example

- The unique solution of recurrence relation $a_{n+1}=da_n$, where $n>=0$, $d$ is a constant and $a_0=A$ is
  - $a_n=Ad^n$, $n>=0$
- Ex 10.1: Solve $a_n=7a_{n-1}$, where $n>=1$ and $a_2=98$
  - $A_0=98/7/7=2 \rightarrow a_n=2*7^n$
- Ex 10.2: A bank pays 6% annual interests, and compounding the interest monthly. If we deposit \$1000, how much will the deposit worth a year later?
  - $p_{n+1}=p_n+0.005p_n$, $p_0=1000$, $p_n=1000*1.005^n$, $p_{12}=1062$

# Converting Nonlinear to Linear

- Ex 10.4: Find $a_{12}$ if $a_{n+1}^2 = 5a_n^2$ , where $a_n > 0$ for $n >= 0$ and $a_0 = 2$
  - The relation is not linear!
  - What if we let $b_n = a_n^2$ ?
  - $b_0 = 4$, $b_n = 4 \cdot 5^n$
  - $b_{12} = 976562500$, $a_{12} = 31250$

# General First-Order Linear Recurrence

- The general form is $a_{n+1}+ca_n=f(n)$, $n>=0$, where $c$ is a constant and $f(n)$ is a function on nonnegative integers
- $F(n)=0$ for all $n$ → homogeneous recurrence
  - Nonhomogeneous, otherwise
- Many techniques are useful for solving nonhomogeneous problems, but non of them can solve all such problems

# Bubble Sort

| i = 1 | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $x_1$ | 7 | | 7 | | 7 | | 7 | j = 2 | 2 |
| $x_2$ | 9 | | 9 | | 9 | j = 3 | 2 | | 7 |
| $x_3$ | 2 | | 2 | j = 4 | 2 | | 9 | | 9 |
| $x_4$ | 5 | j = 5 | 5 | | 5 | | 5 | | 5 |
| $x_5$ | 8 | | 8 | | 8 | | 8 | | 8 |

Four comparisons and two interchanges.

| i = 2 | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x_1$ | 2 | | 2 | | 2 | | 2 |
| $x_2$ | 7 | | 7 | | 7 | j = 3 | 5 |
| $x_3$ | 9 | | 9 | j = 4 | 5 | | 7 |
| $x_4$ | 5 | j = 5 | 5 | | 9 | | 9 |
| $x_5$ | 8 | | 8 | | 8 | | 8 |

Three comparisons and two interchanges.

| i = 3 | | | | | |
|---|---|---|---|---|---|
| $x_1$ | 2 | | 2 | | 2 |
| $x_2$ | 5 | | 5 | | 5 |
| $x_3$ | 7 | | 7 | j = 4 | 7 |
| $x_4$ | 9 | j = 5 | 8 | | 8 |
| $x_5$ | 8 | | 9 | | 9 |

Two comparisons and one interchange.

| i = 4 | | |
|---|---|---|
| $x_1$ | 2 | |
| $x_2$ | 5 | |
| $x_3$ | 7 | |
| $x_4$ | 8 | j = 5 |
| $x_5$ | 9 | |

One comparison but no interchanges.

**Figure 10.3**

# Bubble Sort (cont.)

- Let $a_n$ be the number of comparisons to sort n numbers using bubble sort
  - $a_n = a_{n-1} + (n-1)$, $n >= 2$, $a_1 = 0$
- It is linear first-order, but the term n-1 makes it nonhomogeneous
  - $a_1 = 0$
  - $a_2 = a_1 + (2-1) = 1$
  - $a_3 = a_2 + (3-1) = 1+2$
  - .....
  - In general $a_n = 1+2+\ldots+(n-1) = (n^2-n)/2$

# More Examples

- Ex 10.6: Find the pattern of: 0, 2, 6, 12, 20, 30, 42, …
  - See no pattern, try to compute the difference: 2, 4, 6, 8, 10, 12, … → $a_n - a_{n-1} = 2n,\ n >= 1,\ a_0 = 0$
  - $a_n - a_0 = 2+4+6+\ldots+2n = 2[n(n+1)/2] = n^2+n$
  - Compared against Ex. 9.6
- Ex 10.7 (variable coefficient): Solve the relation $a_n = n * a_{n-1}$, where $n >= 1$ and $a_0 = 1$
  - $a_0 = 1,\ a_1 = 1 * a_0 = 1,\ a_2 = 2 * a_1 = 2,\ ,\ a_3 = 3 * a_2 = 6, \ldots.$
  - In fact, $a_n = n!$

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# Order K Linear Recurrence

- Let $k \in \mathbb{Z}^+, C_0(\neq 0), C_1, \ldots, C_k(\neq 0)$ be real numbers
  - $C_0 a_n + C_1 a_{n-1} + \cdots + C_k a_{n-k} = f(n),\ n \geq k$ is a linear recurrence relation with constant coefficients of order *k*
- If *f(n)=0* for all *n>=0*, the relation is homogeneous, otherwise, it's nonhomogeneous
- We study homogeneous relation of order two in this section
  - $C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} = 0,\ n \geq 2$

# Order 2 Linear Recurrence

- In particular, we look for a solution in the form
  $a_n = cr^n, \; c \neq 0, r \neq 0$

- $C_0 a_n + C_1 a_{n-1} + C_2 a_{n=2} = 0, \; n \geq 2$
  - $C_0 cr^n + C_1 cr^{n-1} + C_2 cr^{n-2} = 0$
  - $C_0 r^2 + C_1 r^1 + C_2 = 0$ ← characteristic equation

- Three cases of the roots $r_1$, $r_2$ ← characteristic roots
  - (a) distinct real numbers
  - (b) complex conjugate pair
  - (c) same real number

# Case A Example 1

- Ex 10.9: Solve recurrence relation $a_n + a_{n-1} - 6a_{n-2} = 0$, where $n>=2$ and $a_0=-1$, $a_1=8$
  - $cr^n+cr^{n-1}-6cr^{n-2}=0$
  - $r^2+r-6=0 \rightarrow r=2, -3$
- Then $a_n=2^n$ or $a_n = (-3)^n$ are two indep. solutions!
- In fact, we can write $a_n=c_1 2^n + c_2 (-3)^n$
- $-1 = c_1+c_2$ and $8=2c_1-3c_2 \rightarrow c_1=1$ and $c_2=-2$
- Solution: $a_n=2^n-2(-3)^n$

# Case A Example 2

- Ex 10.10: Solve the recurrence relation $F_{n+2}=F_{n+1}+F_n$, where $F_0=0$, $F_1=1$
- Let $F_n=cr^n$, we have $r^2-r-1=0$, characteristic roots are $\frac{1\pm\sqrt{5}}{2}$ → let $F_n = c_1(\frac{1+\sqrt{5}}{2})^n + c_2(\frac{1-\sqrt{5}}{2})^n$
- We have $c_1 = \frac{1}{\sqrt{5}}, \ c_2 = -\frac{1}{\sqrt{5}}$
- Solution: $F_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$

# Case A Example 3

- Ex 10.14: Legal arithmetic expressions without parentheses ← 0, 1, 2, …, 9 and +,*,/
- Let $a_n$ be the no. legal expressions with n symbols
  - $a_1$=10, $a_2$=100, but for $n>3$?
  - Case I: if $x$ is an expr. with $n$-$1$ symbols, and the last symbol is a digit. $10a_{n-1}$ way to add a symbol to it
  - Case II: if y is an expr. with n-2 symbols, we have 29 ways to add an operator and a digit to it
  - $a_n=10a_{n-1}+29a_{n-2}$
- Solution: $a_n = \frac{5}{3\sqrt{6}}[(5+3\sqrt{6})^n - (5-3\sqrt{6})^n]$

# Case B Example

- Ex 10.20: Determine $(1+\sqrt{3}i)^{10}$
  - $r = 2,\ \theta = \pi/3$ → $1+\sqrt{3}i = 2(\cos(\pi/3) + i\sin(\pi/3))$
  - We know $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$
  - Hence, $(1+\sqrt{3}i)^{10} = 2^{10}(\cos(4\pi/3) + i\sin(4\pi/3))$

$$= 2^{10}(\frac{-1}{2} - \frac{\sqrt{3}}{2}i) = (-2)^9(1+\sqrt{3}i)$$

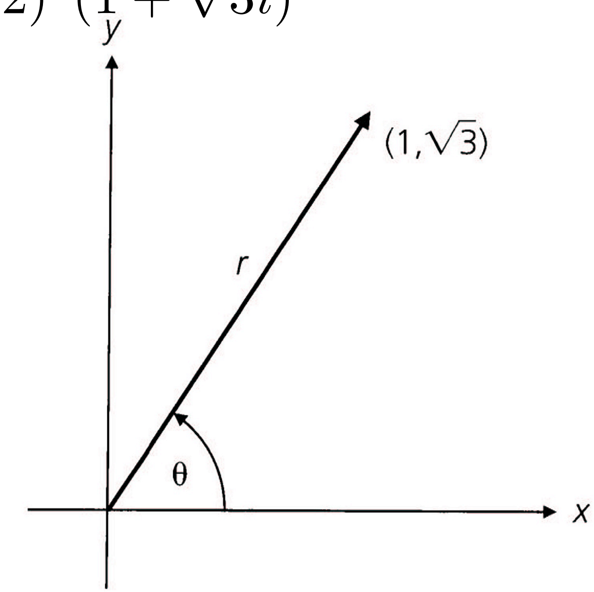


Figure 10.9

# Case B Example (cont.)

- Ex 10.21: Solve $a_n=2a_{n-1}-2a_{n-2},$ where $a_0=1$, $a_1=2$
- Let $a_n=cr^n \rightarrow r^2-2r+2=0 \rightarrow$ roots are $1 \pm i$
- Let $a_n = c_1(1+i)^n + c_2(1-i)^n$
  - $1+i = \sqrt{2}(\cos(\pi/4) + i\sin(\pi/4)),\ 1-i = \sqrt{2}(\cos(\pi/4) - i\sin(\pi/4))$

    $a_n = (\sqrt{2})^n(x_1\cos(n\pi/4) + x_2\sin(n\pi/4)), x_1 = c_1 + c_2, x_2 = (c_1 - c_2)i$

- Solution: $a_n = (\sqrt{2})^n(cos(n\pi/4) + \sin(n\pi/4))$

# Case C Example

- Ex 10.23: Solve $a_{n+2}=4a_{n+1}-4a_n$, $a_0=0$, $a_1=3$
  - Characteristic equation $r^2-4r+4=0 \rightarrow r=2, 2$
  - $2^n$ and $2^n$ are not indep $\rightarrow$ let's try some $g(n)2^n$, where g(n) is not a constant
  - We have $g(n+2)2^{n+2}=4g(n+1)2^{n+1}-4g(n)2^n \rightarrow$ one solution is $g(n)=n$, although there are many other solutions
  - That is, $n2^n$ is another indep. Solution
  - The general solution is then: $a_n=c_1 2^n+c_2 n2^n$
  - With $a_0=1$, $a_1=3$, we have $a_n=2^n+n2^{n-1}$
- Can be generalized to multiple repeated roots

# Outline

# Nonhomogeneous

- We consider the recurrence relations

$$a_0 + C_1 a_{n-1} = f(n), n \geq 1$$
$$a_n + C_1 a_{n-1} + C_2 a_{n-2} = f(n), n \geq 2$$

  - $C_1$, $C_2$ are constant, and f(n) is not zero. $\leftarrow$ nonhomogeneous relations.

- There are no standard way to solve all nonhomogeneous relations, we discuss techniques for certain types of problems

# First Example

- General: Order 1, with $C_1$=-1 → $a_n - a_{n-1} = f(n)$

$$\begin{aligned} a_1 &= a_0 + f(1) \\ a_2 &= a_1 + f(2) = a_0 + f(1) + f(2) \\ \ldots\ldots \\ a_n &= a_{n-1} + f(n) = a_0 + \sum_{i=1}^{n} f(i) \end{aligned}$$

  - We can solve it if we know how to deal with the last term

- Ex 10.25: Solve $a_n$-$a_{n-1}$=$3n^2$, $a_0$=$7$
  - $a_n = a_0 + \sum_{i=1}^{n} f(i) = 7 + 3\sum_{i=1}^{n} i^2 = 7 + \frac{1}{2}(n)(n+1)(2n+1)$

- What is we are not that lucky?

# Undetermined Coefficients

- Method of undetermined coefficients: for both first- and second-order nonhomogeneous relations
  - Rely on solving the associated homogeneous relation
- Let $a_n^{(h)}$ be the general solution of associated homogeneous relation, and $a_n^{(p)}$ be the particular solution to the nonhomogeneous relation
  - $a_n = a_n^{(h)} + a_n^{(p)}$ is the final solution
- We already know how to find $a_n^{(h)}$, to determine $a_n^{(p)}$ we use the form of $f(n)$ to guess a form of $a_n^{(p)}$

# Undetermined Coefficients

- Ex 10.26: Solve $a_n - 3a_{n-1} = 5(7^n)$ , where $n \geq 1, a_0 = 2$
  - The solution to the homogeneous part is $a_n^{(h)} = c(3^n)$
  - $f(n) = 5(7^n)$ →We look for $a_n^{(p)}$ in the form $A(7^n)$
  - That is, $A(7^n) - 3A(7^{n-1}) = 5(7^n)$

    $$\Rightarrow 7A - 3A = 5(7) \Rightarrow A = 35/2$$

    $$\Rightarrow a_n^{(p)} = (35/4)7^n = (5/4)7^{n+1}$$

  - Final solution is $a_n = c(3^n) + (5/4)7^{n+1}$
  - With $a_0 = 2$ , we have c=-27/4

# Another Example

- Ex 10.27: Solve $a_n - 3a_{n-1} = 5(3^n)$, where $n \geq 1, a_0 = 2$
  - Associated homogeneous relation $a_n^{(h)} = c(3^n)$
  - Since $f(n) = 5(3^n)$ , we try $a_n^{(p)} = A(3^n)$ ←but it's not indep. to $a_n^{(h)}$
  - Try $a_n^{(h)} = Bn(3^n)$ instead
  - We have $Bn(3^n) - 3B(n-1)(3^{n-1}) = 5(3^n) \Rightarrow Bn - B(n-1) = 5 \Rightarrow B = 5$
  - The final solution is $a_n = c(3^n) + 5n(3^n)$
  - With $a_0=2$, we have $c=2$

# Generalized Results

- First order: $a_n + C_1 a_{n-1} = kr^n$
  - If $r^n$ is not a solution of the associated homogeneous relation, then $a_n^{(p)} = Ar^n$, where $A$ is a constant
  - Otherwise, $a_n^{(p)} = Bnr^n$, where $B$ is a constant
- Second order: $a_n + C_1 a_{n-1} + C_2 a_{n-2} = kr^n$
  - $a_n^{(p)} = Ar^n$ , if $r^n$ is not a solution of the associated homogeneous relation
  - $a_n^{(p)} = Bnr^n$ , if $a_n^{(h)} = c_1 r^n + c_2 r_1^n$
  - $a_n^{(p)} = Cn^2 r^n$, if $a_n^{(h)} = (c_1 + c_2 n) r^n$

# First Order, Example

- Ex 10.28: Tower of Hanoi with $n$ disks. Let $a_n$ be the minimum number of moves it takes to transfer $n$ disks from peg 1 to peg 3
  - Move n-1 disks from peg 1 to peg 2
  - Move the largest disk from peg 1 to peg 3
  - Move n-1 disks from peg 2 to peg 3
  - Hence, $a_{n+1}=2a_n+1$ and $a_0=0$
  - We know $a_n^{(h)} = c(2^n)$, and $f(n) = 1^n$ is not a solution of the homogeneous relation→we set $a_n^{(p)} = A(1^n) = A$
  - A=2A+1→A=-1→ $a_n = c(2^n) - 1, with\ a_0 = 0 \Rightarrow c = 1$

# Second Order, Example

- Ex 10.34: Solve the recurrence relation $a_{n+2}-4a_{n+1}+3a_n=-200$, $n>=0$, $a_0=3000$, $a_1=3300$
  - $a_n^{(h)} = c_1(3^n) + c_2(1^n)$
  - $f(n)=-100=-100(1^n)$ $\leftarrow$ the same as the solution of the associated homogeneous relation
  - Let $a_n^{(p)} = An \rightarrow A(n+2) - 4A(n+1) + 3An = -200 \Rightarrow A = 100$
  - Hence, $a_n = c_1(3^n) + c_2 + 100n$
  - With $a_0=3000$, $a_1=3300$, $c_1=100$, $c_2=2900$

# Systematic Approach

- Consider $C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} = f(n)$
  - If f(n) is a constant multiple of one of the forms in Table 10.2, and is not a solution of the associated homogeneous relation, then use $a_n^{(p)}$ given in the table

**Table 10.2**

| | $a_n^{(p)}$ |
|---|---|
| $c$, a constant | $A$, a constant |
| $n$ | $A_1 n + A_0$ |
| $n^2$ | $A_2 n^2 + A_1 n + A_0$ |
| $n^t, t \in \mathbf{Z}^+$ | $A_t n^t + A_{t-1} n^{t-1} + \cdots + A_1 n + A_0$ |
| $r^n, r \in \mathbf{R}$ | $Ar^n$ |
| $\sin \theta n$ | $A \sin \theta n + B \cos \theta n$ |
| $\cos \theta n$ | $A \sin \theta n + B \cos \theta n$ |
| $n^t r^n$ | $r^n (A_t n^t + A_{t-1} n^{t-1} + \cdots + A_1 n + A_0)$ |
| $r^n \sin \theta n$ | $Ar^n \sin \theta n + Br^n \cos \theta n$ |
| $r^n \cos \theta n$ | $Ar^n \sin \theta n + Br^n \cos \theta n$ |

# Systematic Approach (cont.)

- Consider $C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} = f(n)$
  - If f(n) is a sum of several terms, and none of them is a solution of the associated homo. relation, then $a_n^{(p)}$ is made up of the sum
  - If part of *f*(*n*), say $f_1(n)$, is a solution of homo. Relation, we find the smallest *s* so that no summand of $n^s f_1(n)$ is solution of the homo. relation. Replace $a_n^{(p)}$ with $n^s(a_n^{(p)})$

# Example

- Ex 10.36: $n$ people at a party, each two persons shakes hands exactly once. Let $a_n$ count the no. handshakes, we have $a_{n+1} = a_n + n,\ n \geq 2,\ a_2 = 1$
  - Intuition, if (n+1)-st person comes, he/she will shake hands with the other n persons
  - By the table, want to try $A_1 n + A_0$ for constants $A_1$, $A_0$
  - But $a_n^{(h)} = c(1^n) = c$, so the $A_0$ term is a solution of the homo. relation → We must multiply $A_1 n + A_0$ by the smallest $n^s$, so that none of the terms is the solution of homo. relation
  - s=1 is sufficient, hence $a_n^{(p)} = A_1 n^2 + A_0 n$

# Example (cont.)

- Ex 10.36: Combine this with $a_{n+1}=a_n+n$ , we have

$$A_1(n+1)^2 + A_0(n+1) = A_1 n^2 + A_0 n + n$$

  - $A_1=1/2, A_0=-1/2$
  - Then, we have $a_n^{(p)} = \frac{1}{2}n^2 + (-\frac{1}{2})n$

    $$a_n = c + \frac{1}{2}(n)(n-1)$$
  - Since $a_2=1 \rightarrow c=0$

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# Order 1 Example

- Ex 10.38: Solve the relation $a_n$-$3a_{n-1}$=$n$, $n$>=$1$, $a_0$=$1$
  - To bring in generating function, we multiply n=1 with x, n=2 with $x^2$, and so on. We have
$$n = 1: \; a_1 x^1 - 3a_0 x^1 = 1x^1$$
$$n = 2: \; a_2 x^2 - 3a_1 x^1 = 2x^2$$
$$\cdots\cdots$$
  - Then we have $\sum_{n=1}^{\infty} a_n x^n - 3\sum_{n=1}^{\infty} a_{n-1} x^n = \sum_{n=1}^{\infty} n x^n$
  - Let f(x) be the ordinary generating function of $a_0$, $a_1$, $a_2$,.., then we have $(f(x) - a_0) - 3x\sum_{n=1}^{\infty} a_{n-1} x^{n-1} = \sum_{n=0}^{\infty} n x^n$
  - And then $(f(x) - 1) - 3xf(x) = \sum_{n=0}^{\infty} n x^n$

# Order 1 Example (cont.)

- Ex 10.38: Solve the relation $a_n$-$3a_{n-1}$=$n$, $n$>=$1$, $a_0$=$1$
  - Recall the generating function of 0, 1, 2, 3, … is $\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + \cdots$
  - Therefore $(f(x)-1) - 3xf(x) = \frac{1}{(1-x)^2}$
  - We write $\frac{x}{(1-x)^2(1-3x)} = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{1-3x}$
  - Solving it we get A=-1/4, B=-1/2, and C=3/4
  - That is: $f(x) = \frac{7/4}{1-3x} + \frac{-1/4}{1-x} + \frac{-1/2}{(1-x)^2}$
  - Using the formulas learned in the generating functions, we have $a_n = \frac{7}{4}3^n - \frac{1}{2}n - \frac{3}{4}$

# Order 2 Example

- Ex 10.39: Solve the relation $a_{n+2}$-$5a_{n+1}+6a_n=2$, $n>=0$, $a_0=3$, $a_1=7$
  - Multiply the relation by $x^{n+2}$ → $a_{n+2}x^{n+2} - 5a_{n+1}x^{n+2} + 6a_nx^{n+2} = 2x^{n+2}$
  - Summation: $\sum_{n=0}^{\infty} a_{n+2}x^{n+2} - 5\sum_{n=0}^{\infty} a_{n+1}x^{n+2} + 6\sum_{n=0}^{\infty} a_nx^{n+2} = 2\sum_{n=0}^{\infty} x^{n+2}$
  - Match the exponents:

$$\sum_{n=0}^{\infty} a_{n+2}x^{n+2} - 5x\sum_{n=0}^{\infty} a_{n+1}x^{n+1} + 6x^2\sum_{n=0}^{\infty} a_nx^n = 2x^2\sum_{n=0}^{\infty} x^n$$

  - Let f(x) be the generating function, we have

$$(f(x) - 3 - 7x) - 5x(f(x) - 3) + 6x^2 f(x) = \frac{2x^2}{1-x}$$

# Order 2 Example (cont.)

- Ex 10.39: Solve the relation $a_{n+2}-5a_{n+1}+6a_n=2$, $n>=0$, $a_0=3$, $a_1=7$
  - Simplify it, we get $f(x) = \dfrac{3-5x}{(1-3x)(1-x)}$
  - Applying partial-fraction decomposition, we have

$$f(x) = \frac{2}{1-3x} + \frac{1}{1-x} = 2\sum_{n=0}^{\infty}(3x)^n + \sum_{n=0}^{\infty} x^n$$

  - Hence, $a_n = 2(3^n) + 1$

# Take-home Exercises

- Exercise 10.1: 2, 3, 7, 9
- Exercise 10.2: 1, 3, 4, 20, 31
- Exercise 10.3: 1, 2, 4, 5, 11
- Exercise 10.4: 1