Midterm #1 Solution)

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Any academic dishonesty will automatically lead to zero point.

- **If the problem asks for reasons of individual steps, you should not omit steps in between. This means, every rule you apply should be a step in the answer, otherwise you will NOT get any points. To make sure you get the full score, try not to omit anything (including straightforward commutative, associative, and double negation rules).**
- **In logics related problems, the final step of your answer has to have the same form as the statement given by the problem. For example, if the problem ask for** $p \vee q$ **and you stop at** $q \vee p$**, you will NOT get any points.**
- **Compound statements is not equal to arguments, when simplifying compound statements, you can only use the laws of logics. When justifying arguments, you can use both the laws of logic and rules of inference.**
- $\overline{A} \cap \overline{B}$ **is considered to be more simplified then** $\overline{(A \cup B)}$**. Give** $\overline{A} \cap \overline{B}$ **as your final answer.**
- **If you wish to use mathematical induction to answer questions, the relation between** $n = k$ **and** $n = k+1$ **is very important, please clearly demonstrate the relation between them. Otherwise you may lose some or even all the points.**
- **Should you have any problems about the guidelines given above, please ask for clarifications in the first 5 minutes of the exam.**

Note: In problem 1, 2, 6 you need to give the reasons, the rules are in the appendices. ***NO*** *partial score will be given if the reasons are not correct/given.*

1) (2%) Give reasons for each step in the simplifications of following compound statement.
$[(p \vee q) \wedge (p \vee \neg q)] \vee q$

Answer (There may be more than one way):

$[(p \vee q) \wedge (p \vee \neg q)] \vee q$

$\Leftrightarrow [p \vee (q \wedge \neg q)] \vee q$ reason: Distributive Laws

$\Leftrightarrow [p \vee F_0] \vee q$ reason: Inverse Laws

$\Leftrightarrow p \vee q$ reason: Identity Laws

2) (2%) Give the reasons for each step to validate the argument. (Hint: proof by contradiction)

$[(p \rightarrow q) \wedge (\neg r \vee s) \wedge (p \vee r)]$

$\therefore \neg q \rightarrow s$

Answer (There may be more than one way):

Here we use proof by contradiction. We assume the conclusion, $\neg q \rightarrow s$ is false, and take it as one of the premises. Then we try to show our premises will derive contradictions.

Steps:

1) $[(p \rightarrow q) \wedge (\neg r \vee s) \wedge (p \vee r)]$ reason: premise
2) $\neg(\neg q \rightarrow s)$ reason: premise (the original conclusion)
3) $\neg(\neg\neg q \vee s)$ reason: 2), $p \rightarrow q \Leftrightarrow \neg p \vee q$
4) $\neg(q \vee s)$ reason: 3), Law of Double Negation
5) $\neg q \wedge \neg s$ reason: 4), DeMorgan's Laws
6) $\neg q$ reason: 5), Rule of Conjunctive Simplification
7) $\neg s$ reason: 5), Rule of Conjunctive Simplification
8) $\neg r \vee s$ reason: 1), Rule of Conjunctive Simplification
9) $s \vee \neg r$ reason: 8), Commutative Laws
10) $\neg r$ reason: 7), 9), Rule of Disjunctive Syllogism
11) $p \vee r$ reason: 1), Rule of Conjunction Simplification
12) $r \vee p$ reason: 11), Commutative Laws
13) p reason: 10), 12), Rule of Disjuctive Syllogism
14) $p \rightarrow q$ reason: 1), Rule of Conjunction Simplification
15) q reason: 13), 14), Rule of Detachment
16) $q \wedge \neg q$ reason: 6), 15), Rule of Conjunction Simplification
17) F_0 reason: 16), Inverse Laws

Based on the premises, in the end we get a F_0, which means a contradiction. This happens because there is an inconsistency among the premises. In this case, it's the assumption we made. Therefore our original assumption is wrong, $\neg q \rightarrow s$ is true.

3) (2%) Use truth tables to verify that

$(p \vee q) \to [q \to (p \wedge q)]$ is a NOT tautology.

Answer:

When $p = 0$ and $q = 1$, $(p \vee q) \to [q \to (p \wedge q)] = 0$, therefore it is not a tautology.

p	q	$p \vee q$	$p \wedge q$	$q \to (p \wedge q)$	$(p \vee q) \to [(q \to (p \wedge q)]$
0	0	0	0	1	1
0	1	1	0	0	0
1	0	1	0	1	1
1	1	1	1	1	1

4) (2%) Let $p(x)$, $q(x)$, and $r(x)$ be the following open statements,

$p(x) : x^2 - 7x + 10 = 0$

$q(x) : x^2 - 2x - 3 = 0$

$r(x) : x < 0$

Determine the truth or falsity of the following statements, where the universe is all integers. If a statement is false, provide a counterexample or explanation.

a) $\forall x[p(x) \rightarrow \neg r(x)]$
b) $\forall x[q(x) \rightarrow r(x)]$
c) $\exists x[q(x) \rightarrow r(x)]$
d) $\exists x[p(x) \rightarrow r(x)]$

Answer:

a) T
b) F, when $x = 3$, $q(3)$ is true, but $r(3)$ is false.
c) T
d) T

5) (2%) Let $A = \{1, 2, \{2\}\}$, which of the following statements are true?

a) $1 \in A$
b) $\{1\} \in A$
c) $\{1\} \subseteq A$
d) $\{\{1\}\} \subseteq A$
e) $\{2\} \in A$
f) $\{2\} \subseteq A$
g) $\{\{2\}\} \subseteq A$
h) $\{\{2\}\} \subset A$

Answer:

a) T
b) F
c) T
d) F
e) T
f) T
g) T
h) T

6) (2%) Using the laws of set theory, simplify each of the following:

a) $(A \cap B) \cup (A \cap B \cap \overline{C} \cap D) \cup (\overline{A} \cap B)$

b) $(A - B) \cup (A \cap B)$

Answer (There may be more than one answer, in (a) we list two kinds of answer):

($a.1$)

$\Leftrightarrow (A \cap B) \cup (A \cap B \cap \overline{C} \cap D) \cup (\overline{A} \cap B)$

$\Leftrightarrow (A \cap B) \cup (\overline{A} \cap B) \cup (A \cap B \cap \overline{C} \cap D)$ reason: Commutative Laws

$\Leftrightarrow (B \cap A) \cup (B \cap \overline{A}) \cup (A \cap B \cap \overline{C} \cap D)$ reason: Commutative Laws

$\Leftrightarrow B \cap (A \cup \overline{A}) \cup (A \cap B \cap \overline{C} \cap D)$ reason: Distributive Laws

$\Leftrightarrow B \cap \mathbb{U} \cup (A \cap B \cap \overline{C} \cap D)$ reason: Inverse Laws

$\Leftrightarrow B \cup (A \cap B \cap \overline{C} \cap D)$ reason: Identity Laws

$\Leftrightarrow B \cup (B \cap A \cap \overline{C} \cap D)$ reason: Commutative Laws

$\Leftrightarrow B$ reason: Absorption Laws (B as p, $A \cap \overline{C} \cap D$ as q)

($a.2$)

$\Leftrightarrow (A \cap B) \cup (A \cap B \cap \overline{C} \cap D) \cup (\overline{A} \cap B)$

$\Leftrightarrow (A \cap B) \cup (\overline{A} \cap B)$ reason: Absorption Laws ($A \cap B$ as p, $\overline{C} \cap D$ as q)

$\Leftrightarrow (A \cup \overline{A}) \cap B$ reason: Distributive Laws

$\Leftrightarrow \mathbb{U} \cap B$ reason: Inverse Laws

$\Leftrightarrow B$ reason: Identity Laws

(b)

$(A - B) \cup (A \cap B)$

$\Leftrightarrow (A \cap \overline{B}) \cup (A \cap B)$ reason: Definition of $(A - B)$

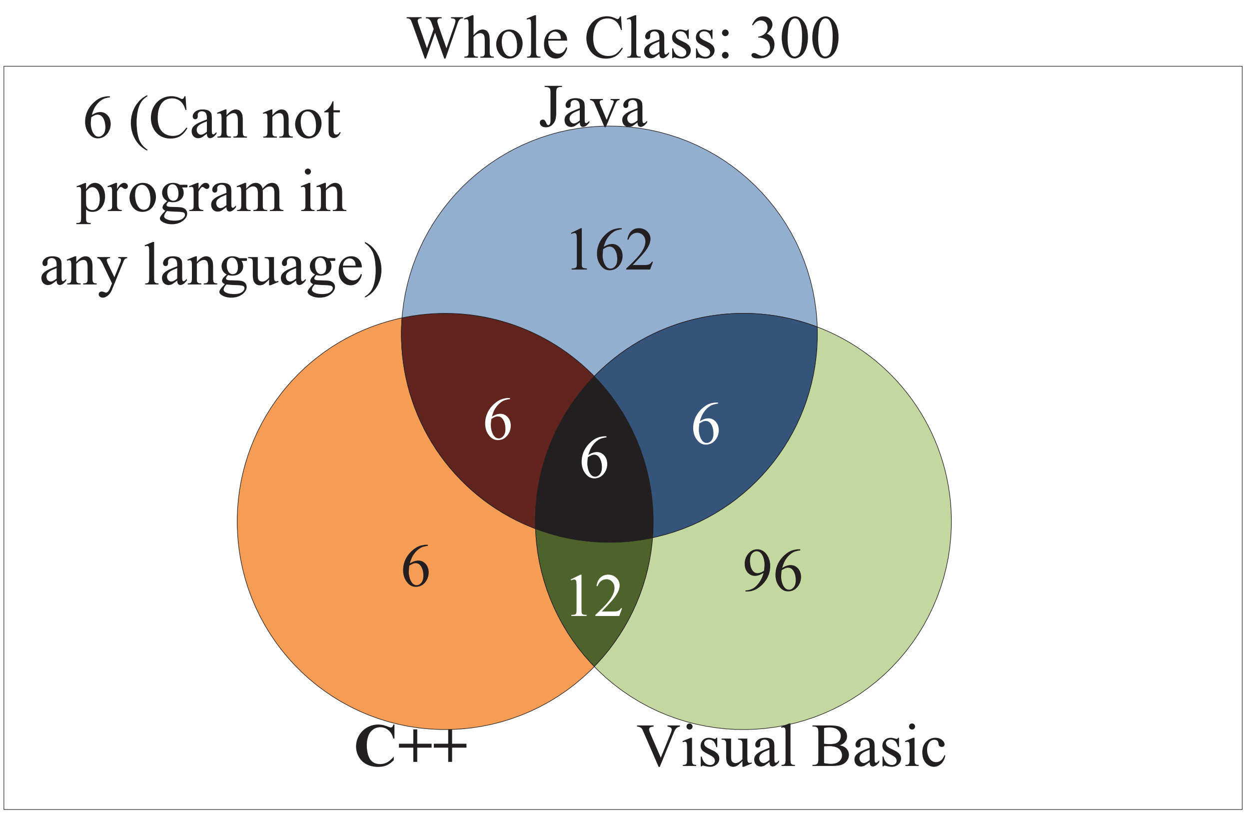
$\Leftrightarrow A \cap (\overline{B} \cup B)$ reason: Distributive Laws

$\Leftrightarrow A \cap \mathbb{U}$ reason: Inverse Laws

$\Leftrightarrow A$ reason: Identity Laws

7) (2%) The freshman class of a private engineering college has 300 students. It is known that 180 can program in Java, 120 in Visual BASIC, 30 in C++, 12 in Java and C++, 18 in Visual BASIC and C++, 12 in Java and Visual BASIC, and 6 in all three languages. A student is selected at random.

   a) What is the probability that he/she can program in exactly one language?
   b) What is the probability that he/she can program in more than one language?



Answer:

a) $\frac{(162+96+6)}{300} = \frac{264}{300} = \frac{22}{25}$

b) $\frac{(6+6+6+12)}{300} = \frac{30}{300} = \frac{1}{10}$

8) (2%) For $k \in \mathbb{Z}^+$ verify that $k^2 = \binom{k}{2} + \binom{k+1}{2}$.

Answer:

Basis: For $k = 1$, $1^2 = \binom{1}{2} + \binom{2}{2} = 0 + 1 = 1$

$\Rightarrow$ Holds when $k = 1$.

Assume when $k = n$, $k^2 = \binom{k}{2} + \binom{k+1}{2}$ holds.

When $k = n + 1$,

$\Rightarrow (n+1)^2 = n^2 + 2n + 1$

$= (\binom{n}{2} + \binom{n+1}{2}) + 2n + 1$

$= \binom{n+1}{2} + (\binom{n}{2} + 2n + 1)$

$= \binom{n+1}{2} + (\frac{n(n-1)}{2} + 2n + 1)$

$= \binom{n+1}{2} + (\frac{n^2+3n+2}{2})$

$= \binom{n+1}{2} + (\frac{(n+2)(n+1)}{2})$

$= \binom{n+1}{2} + \binom{n+2}{2}$

$= \binom{n+1}{2} + \binom{(n+1)+1}{2}$

$\Rightarrow$ When $k = n$ holds, the assumption also holds for $k = n + 1$,

by the priciple of Mathematical Induction,

For all k in $\mathbb{Z}^+$, $k^2 = \binom{k}{2} + \binom{k+1}{2}$ holds.

9) (3%) Consider four-bit two's complement representation.

   a) Convert $7, 2, 3, -1$ into four-bit two's complement representation.
   b) Solve $7 + 2$
   c) Solve $3 + (-1)$
   d) Using your own word, briefly explain what is overflow error. Is there any overflow error in (b) and (c)?

Answer:

a) $7 = 0111, 2 = 0010, 3 = 0011, -1 = 1111$

b) 0111

   0010

   $1001 \Rightarrow$ overflow, the answer should be 9 while the result is -7.

c) 0011

   1111

   $0010 = 2$

d) When the the value of the number exceeds the limit imposed by the number of bits, overflow error happens.

   Resulting incorrect outcome in arithmetic operation. In (b) there is an overflow error, the summation of two positive value gives a negative outcome.

10) (2%) For each of the following functions $f: \mathbb{R} \rightarrow \mathbb{R}$, determine whether the function is one-to-one and whether it is onto. If the function is not onto, determine the range $f(\mathbb{R})$.

a) $f(x) = x^2$
b) $f(x) = x + 7$
c) $f(x) = 2x - 3$
d) $f(x) = x^3$

Answer:

a) not one-to-one, not onto, $f(\mathbb{R}) = [0, +\infty)$.
b) one-to-one, onto.
c) one-to-one, onto.
d) one-to-one, onto.

11) (2%) Let $A = \{0, 1, 2\}$ and $B = \{0, 1, 2, 3, 4, 5, 6\}$.

a) How many functions are there from B to A?

b) How many functions in (a) are onto?

Answer:

a) $3^7 = 2187$

b) $(3!) * S(7, 3) = 6 * 301 = 1806$

12) (2%) Let $A = \{a, b, c, d, e, z\}$.

a) How many closed binary operations on A are commutative?

b) How many closed binary operations on A have z as the identity?

Answer:

a) 6^{21}

b) 6^{25}

- **Appendix A: The Laws of Logic**

For any primitive statements p, q, r, any tautology T_0, and any contradiction F_0.

1)	$\neg\neg p \Leftrightarrow p$	Law of *Double Negation*
2)	$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$	*DeMorgan's* Laws
	$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$	
3)	$p \vee q \Leftrightarrow q \vee p$	*Commutative* Laws
	$p \wedge q \Leftrightarrow q \wedge p$	
4)	$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$	*Associative* Laws
	$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$	
5)	$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$	*Distributive* Laws
	$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$	
6)	$p \vee p \Leftrightarrow p$	*Idempotent* Laws
	$p \wedge p \Leftrightarrow p$	
7)	$p \vee F_0 \Leftrightarrow p$	*Identity* Laws
	$p \wedge T_0 \Leftrightarrow p$	
8)	$p \vee \neg p \Leftrightarrow T_0$	*Inverse* Laws
	$p \wedge \neg p \Leftrightarrow F_0$	
9)	$p \vee T_0 \Leftrightarrow T_0$	*Domination* Laws
	$p \wedge F_0 \Leftrightarrow F_0$	
10)	$p \vee (p \wedge q) \Leftrightarrow p$	*Absorption* Laws
	$p \wedge (p \vee q) \Leftrightarrow p$	
11)	$p \rightarrow q \Leftrightarrow \neg p \vee q$	$p \rightarrow q \Leftrightarrow \neg p \vee q$ (write this as the reason)

- **Appendix B: Rules of Inference**

Table 2.19

Rule of Inference	Related Logical Implication	Name of Rule
1) p $p \rightarrow q$ $\therefore q$	$[p \wedge (p \rightarrow q)] \rightarrow q$	Rule of Detachment (Modus Ponens)
2) $p \rightarrow q$ $q \rightarrow r$ $\therefore p \rightarrow r$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$	Law of the Syllogism
3) $p \rightarrow q$ $\neg q$ $\therefore \neg p$	$[(p \rightarrow q) \wedge \neg q] \rightarrow \neg p$	Modus Tollens
4) p q $\therefore p \wedge q$		Rule of Conjunction
5) $p \vee q$ $\neg p$ $\therefore q$	$[(p \vee q) \wedge \neg p] \rightarrow q$	Rule of Disjunctive Syllogism
6) $\neg p \rightarrow F_0$ $\therefore p$	$(\neg p \rightarrow F_0) \rightarrow p$	Rule of Contradiction
7) $p \wedge q$ $\therefore p$	$(p \wedge q) \rightarrow p$	Rule of Conjunctive Simplification
8) p $\therefore p \vee q$	$p \rightarrow p \vee q$	Rule of Disjunctive Amplification
9) $p \wedge q$ $p \rightarrow (q \rightarrow r)$ $\therefore r$	$[(p \wedge q) \wedge [p \rightarrow (q \rightarrow r)]] \rightarrow r$	Rule of Conditional Proof
10) $p \rightarrow r$ $q \rightarrow r$ $\therefore (p \vee q) \rightarrow r$	$[(p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow [(p \vee q) \rightarrow r]$	Rule for Proof by Cases
11) $p \rightarrow q$ $r \rightarrow s$ $p \vee r$ $\therefore q \vee s$	$[(p \rightarrow q) \wedge (r \rightarrow s) \wedge (p \vee r)] \rightarrow (q \vee s)$	Rule of the Constructive Dilemma
12) $p \rightarrow q$ $r \rightarrow s$ $\neg q \vee \neg s$ $\therefore \neg p \vee \neg r$	$[(p \rightarrow q) \wedge (r \rightarrow s) \wedge (\neg q \vee \neg s)] \rightarrow (\neg p \vee \neg r)$	Rule of the Destructive Dilemma

- **Appendix C: The Laws of Set Theory**

For any sets A, B, and C taken from a universe $\mathbb{U}$

1)	$\overline{\overline{A}} = A$	Laws of *Double Complement*
2)	$\overline{A \cup B} = \overline{A} \cap \overline{B}$	*DeMorgan's* Laws
	$\overline{A \cap B} = \overline{A} \cup \overline{B}$	
3)	$A \cup B = B \cup A$	*Commutative* Laws
	$A \cap B = B \cap A$	
4)	$A \cup (B \cup C) = (A \cup B) \cup C$	*Associative* Laws
	$A \cap (B \cap C) = (A \cap B) \cap C$	
5)	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	*Distributive* Laws
	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	
6)	$A \cup A = A$	*Idempotent* Laws
	$A \cap A = A$	
7)	$A \cup \varnothing = A$	*Identity* Laws
	$A \cap \mathbb{U} = A$	
8)	$A \cup \overline{A} = \mathbb{U}$	*Inverse* Laws
	$A \cap \overline{A} = \varnothing$	
9)	$A \cup \mathbb{U} = \mathbb{U}$	*Domination* Laws
	$A \cap \varnothing = \varnothing$	
10)	$A \cup (A \cap B) = A$	*Absorption* Laws
	$A \cap (A \cup B) = A$	