Name: Student ID:

Quiz #10 6%

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This is a closed book test. Any academic dishonesty will automatically lead to zero point. If the total points are more than 6 points, you will get at most 6 points out of it.

1) (2%) If Laura invests $\$100$ at 1.5% interest compounded quarterly (every three months), how many months must she wait for her money to double? (She cannot withdraw the money before the quarter is up.)

   Hint: $\log_{10} 2 = 0.301029$, $\log_{10} 1.015 = 0.006466$

   Answer:

   $P_n = 100(1 + 0.015)^n, P_0 = 100$

   $200 \leq 100(1.015)^n \Rightarrow 2 \leq (1.015)^n$

   $\Rightarrow \log_{10} 2 \leq n \log_{10} 1.015$

   $n \geq 46.55$

   $n = 47$

   Hence, Laura must wait $(47)(3) = 141$ months for her money to double.

2) (2%) Find and solve a recurrence relation for the number of ways to park motorcycles and compact cars in a row of n spaces if each cycle requires one space and each compact car needs two. (Motorcycles comes in two distinct models, compact cars have only one model. We want to use up all the n spaces.)

   Answer:

   Here $a_0 = 1$ and $a_1 = 2$, For $n \geq 2$, consider the *nth* space. If the space is occupied by a motorcycle – in one of two ways, then we have $2a_{n-1}$ of the ways to fill the n spaces. Further, there are a_{n-2} ways to fill the n spaces when a compact car occupies positions $n-1$ and n. These two cases are exhaustive and have nothing in common, so

   $a_n = 2a_{n-1} + a_{n-2}, n \geq 2, a_0 = 1, a_1 = 2.$

Under the substitution we have $r^2 - 2r - 1 = 0$, so $r = 1 \pm \sqrt{2}$ and $a_n = c_1(1+\sqrt{2})^n + c_2(1-\sqrt{2})^n$, $n \geq 0$. From $1 = a_0 = c_1 + c_2$ and $2 = a_1 = c_1(1+\sqrt{2}) + c_2(1-\sqrt{2})$, we have $c_1 = \frac{2+\sqrt{2}}{4}$, and $c_2 = \frac{2-\sqrt{2}}{4}$. So $a_n = ((\sqrt{2}+2)/4)(1+\sqrt{2})^n + ((2-\sqrt{2})/4)(1-\sqrt{2})^n = (\frac{1}{2\sqrt{2}})[(1+\sqrt{2})^{n+1} - (1-\sqrt{2})^{n+1}], n \geq 0$.

3) (2%) Solve each of the following recurrence relations.

a) $a_{n+1} - a_n = 2n + 3, n \geq 0, a_0 = 1$

b) $a_{n+1} - 2a_n = 5, n \geq 0, a_0 = 1$

Answer:

a) $a_1 = a_0 + 0 + 3$

$a_2 = a_1 + 2 + 3 = a_0 + 2 + 2(3)$

$a_3 = a_2 + 2(2) + 3 = a_0 + 2 + 2(2) + 3(3)$

$a_4 = a_3 + 2(3) + 3 = a_0 + [2 + 2(2) + 2(3)] + 4(3)$

$\vdots$

$a_n = a_0 + 2[1 + 2 + 3 + \cdots + (n-1)] + n(3) = 1 + 2[n(n-1)/2] + 3n$

$= 1 + n(n-1) + 3n = n^2 + 2n + 1 = (n+1)^2, n \geq 0$

b) $a_1 = 2a_0 + 5 = 2 + 5$

$a_2 = 2a_1 + 5 = 2^2 + 2 \cdot 5 + 5$

$a_3 = 2a_2 + 5 = 2^3 + (2^2 + 2 + 1)5$

$\vdots$

$a_n = 2^n + 5(1 + 2 + 2^2 + \cdots + 2^{n-1}) = 2^n + 5(2^n - 1) = 6(2^n) - 5, n \geq 0$

4) (2%) Solve the following recurrence relations. (No final answer should involve complex numbers.)

a) $a_n = 5a_{n-1} + 6a_{n-2}, n \geq 2, a_0 = 1, a_1 = 3$

b) $a_{n+2} + a_n = 0, n \geq 0, a_0 = 0, a_1 = 3$

Answer:

a) Let $a_n = cr_n, c, r \neq 0$. Then the characteristic equation is $r^2 - 5r - 6 = 0 = (r-6)(r+1)$, so $r = -1, 6$ are the characteristic roots.

$a_n = A(-1)^n + B(6)^n$

$1 = a_0 = A + B$

$3 = a_1 = -A + 6B$, so $B = 4/7$ and $A = 3/7$. $a_n = (3/7)(-1)^n + (4/7)(6)^n$, $n \geq 0$.

b) With $a_n = cr^n$, $c, r \neq 0$, the characteristic equation $r^2+1 = 0$ yields the characteristic roots $\pm i$.

Hence $a_n = A(i)^n + B(-i)^n = A(\cos(\pi/2) + i\sin(\pi/2))^n + B(\cos(\pi/2) + i\sin(-\pi/2))^n = C\cos(n\pi/2) + D\sin(n\pi/2)$.

$0 = a_0 = C, 3 = a_1 = D\sin(\pi/2) = D$, so $a_n = 3\sin(n\pi/2), n \geq 0$.