Department of Computer Science
National Tsing Hua University

CS 2336: Discrete Mathematics

Chapter 2

Fundamentals of Logic

Instructor: Cheng-Hsin Hsu

Outline

2.1 Basic Connectives and Truth Tables

2.2 Logical Equivalence: The Laws of Logic

2.3 Logic Implication: Rules of Inference

2.4 The Use of Quantifiers

2.5 Quantifiers, Definitions, and Proofs of Theorems

Statements

- Statements (or propositions): declarative sentences that are either true or false, but not both
  - To make assertions ← building blocks of mathematical theory

- Examples of statements
  - p: Combinatorics is mandatory for freshmen
  - q: J. K. Rowling wrote Harry Potter
  - r: $2 + 3 = 5$

Statements (cont.)

- Sentences without truth values are not statements
  - Exclamation: What a beautiful afternoon!
  - Command: Get up and do your exercises.

- Primitive statement: statement that cannot be broken down into simpler forms
- Compound statement: negation or combination of two or more statements using logical connectives (details in the following slides)

Negation

- One way to transform a given statement p is its negation
  - $\neg p$, which is read as "not p"
  - No longer a primitive statement
- Examples
  - p : Combinatorics is mandatory for freshmen
  - $\neg p$: Combinatorics is not mandatory for freshmen

Logical Connectives

- Conjunction: $p \wedge q$, which is read "*p* and *q*"
- Disjunction: $p \vee q$, which is read "*p* or *q*"
- Disjunction: $p \veebar q$, which is read "*p* exclusive or *q*"
- Implication: $p \to q$, can be read as "*p* implies *q*"
  - "if *p*, then *q*", "*p* only if *q*"
  - "*p* is sufficient for *q*", "*p* is a sufficient condition for *q*"
  - "*q* is necessary for *p*", "*q* is a necessary condition for *p*"
- Biconditional: $p \leftrightarrow q$, which is read "*p* if and only if *q*" or "*p* is necessary and sufficient for *q*"

Truth Tables

- 0 → False, 1 → True
- Implication: If “2+3 = 6”, then “2+4 =7”

Table 2.1

p	$\neg p$
0	
1	

Table 2.2

p	q	$p \wedge q$	$p \vee q$	$p \veebar q$	$p \rightarrow q$	$p \leftrightarrow q$
0	0					
0	1					
1	0					
1	1					

Example 2.1

- Primitive statement
  - s: Phyllis goes out for a walk
  - t: The moon is out
  - u: It is snowing
- Map compound statements to English
  - $(t \wedge \neg u) \rightarrow s$: If the moon is out and it's not snowing, then Phyllis goes out for a walk
  - $t \rightarrow (\neg u \rightarrow s)$: If the moon is out, then if it's not snowing Phyllis goes out for a walk
  - $\neg(s \leftrightarrow (u \vee t))$: It is not the case that Phyllis goes out for a walk if and only if it is snowing or the moon is out

Example 2.1 (cont.)

- Primitive statement
  - s: Phyllis goes out for a walk
  - t: The moon is out
  - u: It is snowing
- Map English to compound statements
  - Phyllis will go out walking iff the moon is out: $s \leftrightarrow t$
  - If it is snowing and the moon is not out, then Phyllis will go out for a walk: $(u \wedge \neg t) \rightarrow s$
  - It is snowing but Phyllis will still go out for a walk: $u \wedge s$

Example 2.5

- Develop the following truth table

Table 2.4

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$(p \vee q) \wedge r$
0	0	0				
0	0	1				
0	1	0				
0	1	1				
1	0	0				
1	0	1				
1	1	0				
1	1	1				

- Why should we avoid $p \vee q \wedge r$?

Example 2.6

Table 2.5

p	q	$p \vee q$	$p \to (p \vee q)$	$\neg p$	$\neg p \wedge q$	$p \wedge (\neg p \wedge q)$
0	0	0	1	1	0	0
0	1	1	1	1	1	0
1	0	1	1	0	0	0
1	1	1	1	0	0	0

- Observations
  - $p \to (p \vee q)$: is always true
  - $p \to (\neg p \wedge q)$: is always false

Definition 2.1

- **Tautology: denoted as** T_0
  - a compound statement that is true for all truth value assignments
- **Contradiction: denoted as** F_0
  - a compound statement is false for all truth value assignments

Outline

2.1 Basic Connectives and Truth Tables

2.2 Logical Equivalence: The Laws of Logic

2.3 Logic Implication: Rules of Inference

2.4 The Use of Quantifiers

2.5 Quantifiers, Definitions, and Proofs of Theorems

Equivalence

- How can we determine two statements are equivalent?
  - Algebra of logics: we use truth table to check whether two statements are the same
- Example 2.7

Table 2.6

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	0	1	1

Definition 2.2

- Two statements s_1 and s_2 are *logically equivalent*, and written as $s_1 \iff s_2$, when the statement s_1 is true (false) *if and only if* the statement s_2 is true (false)

- Example 2.7: $\neg p \vee q \iff (p \to q)$
  - Based on truth values for all possible choices
  - One way to get rid of $\to$ connectives

Some Examples

- $(p \leftrightarrow q) \iff (p \to q) \wedge (q \to p)$
- $(p \leftrightarrow q) \iff (\neg p \vee q) \wedge (\neg q \vee p)$:
  - One way to get rid of $\leftrightarrow$ connectives

Table 2.7

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$	$p \leftrightarrow q$
0	0	1	1	1	1
0	1	1	0	0	0
1	0	0	1	0	0
1	1	1	1	1	1

De Morgan's Laws

- Example 2.8
  - $\neg(p \wedge q) \iff \neg p \vee \neg q$
  - $\neg(p \vee q) \iff \neg p \wedge \neg q$

Table 2.9

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$p \vee q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
0	0	0	1	1	1	1	0	1	1
0	1	0	1	1	0	1	1	0	0
1	0	0	1	0	1	1	1	0	0
1	1	1	0	0	0	0	1	0	0

Distributive Law

- Example 2.9
  - $p \wedge (q \vee r) \iff (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \iff (p \vee q) \wedge (p \vee r)$

Table 2.10

p	q	r	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r)$	$(p \vee q) \wedge (p \vee r)$
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	0	0	0	0
0	1	1	0	0	1	1
1	0	0	0	0	1	1
1	0	1	1	1	1	1
1	1	0	1	1	1	1
1	1	1	1	1	1	1

The Laws of Logic

- Law of double negation $\neg\neg p \iff p$
- DeMorgan's laws $\neg(p \vee q) \iff \neg p \wedge \neg q$

  $\neg(p \wedge q) \iff \neg p \vee \neg q$
- Commutative laws $p \vee q \iff q \vee p$

  $p \wedge q \iff q \wedge p$
- Associate laws $p \vee (q \vee r) \iff (p \vee q) \vee r$

  $p \wedge (q \wedge r) \iff (p \wedge q) \wedge r$
- Distributive laws $p \vee (q \wedge r) \iff (p \vee q) \wedge (p \vee r)$

  $p \wedge (q \vee r) \iff (p \wedge q) \vee (p \wedge r)$

The Laws of Logic

- Idempotent laws $p \vee p \iff p$

  $p \wedge p \iff p$

- Identity laws $p \vee F_0 \iff p$

  $p \wedge T_0 \iff p$

- Inverse laws $p \vee \neg p \iff T_0$

  $p \wedge \neg p \iff F_0$

- Domination laws $p \vee T_0 \iff T_0$

  $p \wedge F_0 \iff F_0$

- Absorption laws $p \vee (p \wedge q) \iff p$

  $p \wedge (p \vee q) \iff p$

Definition 2.3

- Let s be a statement with only connectives $\wedge$ and $\vee$, the dual of s, denoted s^d, is the statement derived from s by replacing $\wedge$ and $\vee$ with $\vee$ and $\wedge$, respectively, and T_0 and F_0 with F_0 and T_0 , respectively
- Example:
  - $s : (p \wedge \neg q) \wedge (r \wedge T_0)$
  - $s^d : (p \vee \neg q) \vee (r \vee F_0)$

Principe of Duality

- Theorem 2.1: Let *s* and *t* be statements containing no logical connectives other than $\vee$ and $\wedge$. If $s \iff t$ then $s^d \iff t^d$

- Then, each law can be proved by showing only one of the laws in each pair

Substitution Rules

- Supposing a compound statement P is a tautology, and p is a primitive statement. Replacing all occurrences of p in P by the same statement q results in another tautology
- Let P be a compound statement, which consists of a statement p. Assuming statement $q \iff p$. We generate P' by replacing some p in P with q. We have $P \iff P'$

Example 2.10

Show $\neg[(r \wedge s) \vee (t \rightarrow u)] \iff [\neg(r \wedge s) \wedge \neg(t \rightarrow u)]$

Proof:

By DeMorgan's Laws, the following statement is a tautology

$$P0 : \neg[p \vee q] \leftrightarrow [\neg p \wedge \neg q]$$

Since *p* is a primitive statement, we may replace it with $r \wedge s$, which leads to another tautology

$$P1 : \neg[(r \wedge s) \vee q] \leftrightarrow [\neg(r \wedge s) \wedge \neg q]$$

We then replace q with $t \rightarrow u$, which gives us another tautology

$$P2 : \neg[(r \wedge s) \vee (t \rightarrow u)] \leftrightarrow [\neg(r \wedge s) \wedge \neg(t \rightarrow u)]$$

Consider the LHS and RHS as two statements, yield the logical equivalence.

Example 2.16

Simplify the compound statement $(p \vee q) \wedge \neg(\neg p \wedge q)$

sol: Reasons

$\Longleftrightarrow$	$(p \vee q) \wedge (\neg\neg p \vee \neg q)$	DeMorgan
$\Longleftrightarrow$	$(p \vee q) \wedge (p \vee \neg q)$	Double negation
$\Longleftrightarrow$	$p \vee (q \wedge \neg q)$	Distributive law
$\Longleftrightarrow$	$p \vee F_0$	Inverse law
$\Longleftrightarrow$	p	Identify law

Thus, we have

$$(p \vee q) \wedge \neg(\neg p \wedge q) \iff p$$

Example 2.17

Simplify the compound statement $\neg[\neg[(p \vee q) \wedge r] \vee \neg q]$

sol:	Reasons
$\Longleftrightarrow \neg\neg[(p \vee q) \wedge r] \wedge \neg\neg q$	DeMorgan
$\Longleftrightarrow [(p \vee q) \wedge r] \wedge q$	Double negation
$\Longleftrightarrow (p \vee q) \wedge (r \wedge q)$	Associative law
$\Longleftrightarrow (p \vee q) \wedge (q \wedge r)$	Commutative law
$\Longleftrightarrow [(p \vee q) \wedge q] \wedge r$	Associative law
$\Longleftrightarrow q \wedge r$	Absorption law

Outline

2.1 Basic Connectives and Truth Tables

2.2 Logical Equivalence: The Laws of Logic

2.3 Logic Implication: Rules of Inference

2.4 The Use of Quantifiers

2.5 Quantifiers, Definitions, and Proofs of Theorems

Argument

- Consider a general implication $(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow q$,

where $p_1, p_2, \ldots, p_n$ are premises and statement q is the conclusion of this argument

- This argument is valid if whenever all premises are true, then the conclusion is also true
  - If any of the premises is false, then the implication is automatically true
  - A more systematic way to show an argument is valid is to show it is tautology!

Example 2.19

- Consider three primitive statements defined as
  - p: Roger studies
  - q: Roger plays basketball
  - r: Roger passes discrete mathematics
- Define three premises as
  - p_1: If roger studies, then he will pass the course
  - p_2: If Roger doesn't play basketball, then he will study
  - p_3: Roger failed the course
- Determine whether the following argument is valid

$$(p_1 \wedge p_2 \wedge p_3) \rightarrow q$$

Example 2.19 (cont.)

- Rewrite the premises as
  - p_1: $p \to r$
  - p_2: $\neg q \to p$
  - p_3: $\neg r$
- What we really want to show is

$$[(p \to r) \wedge (\neg q \to p) \wedge \neg r] \to q$$

- We check the truth table, and found the above statement is a tautology, hence it is a valid argument

Example 2.19 (cont.)

Table 2.14

			p_1	p_2	p_3	$(p_1 \wedge p_2 \wedge p_3) \rightarrow q$
p	q	r	$p \rightarrow r$	$\neg q \rightarrow p$	$\neg r$	$[(p \rightarrow r) \wedge (\neg q \rightarrow p) \wedge \neg r] \rightarrow q$
0	0	0	1	0	1	1
0	0	1	1	0	0	1
0	1	0	1	1	1	1
0	1	1	1	1	0	1
1	0	0	0	1	1	1
1	0	1	1	1	0	1
1	1	0	0	1	1	1
1	1	1	1	1	0	1

Definition 2.4

- For any two statements *p* and *q*, if $p \to q$ is a tautology, then *p* logically implies *q* and we write $p \implies q$
  - $p \to q$ is called logic implication, and is a tautology
  - *q* is true whenever *p* is true
- If $p \iff q$, then $p \implies q$ and $q \implies p$
- If $p \implies q$ and $q \implies p$, then $p \iff q$
- $p \not\Rightarrow q$ denotes *p* does not logically implies *q*

How to Establish Validity?

- So far we only know how to do this using truth tables → so many rows to check for tautology
- A simple optimization: let's focus on the rows with 1 on the LHS.

Table 2.14

			p_1	p_2	p_3	$(p_1 \wedge p_2 \wedge p_3) \rightarrow q$
p	q	r	$p \rightarrow r$	$\neg q \rightarrow p$	$\neg r$	$[(p \rightarrow r) \wedge (\neg q \rightarrow p) \wedge \neg r] \rightarrow q$
0	0	0	1	0	1	1
0	0	1	1	0	0	1
0	1	0	1	1	1	1
0	1	1	1	1	0	1
1	0	0	0	1	1	1
1	0	1	1	1	0	1
1	1	0	0	1	1	1
1	1	1	1	1	0	1

Efficiently Validate Arguments

- Construct a list of techniques, called rules of inference, to validate arguments without truth tables
  - Specifically check what if the premises have value 1, without building truth tables
  - Step-by step validations of $(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow q$

- We derive a few interference rules in the following

Rule of Detachment

- Example 2.22: $[p \wedge (p \rightarrow q)] \rightarrow q$, which can be validated using a truth table

Table 2.16

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$	$[p \wedge (p \rightarrow q)] \rightarrow q$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

- Tabular form

$$\begin{array}{ll} & p \\ & p \rightarrow q \\ \hline \therefore & q \end{array}$$

premises

therefore → conclusion

Rule of Detachment (cont.)

- Example 1
  - p1) Lydia wins a ten-million-dollar lottery
  - p2) If Lydia wins a ten-million-dollar lottery, then Kay will quit his job
  - c) Therefore, Kay quit his job

$$\begin{array}{rl} & p \\ & p \rightarrow q \\ \hline \therefore & q \end{array}$$

- Example 2
  - p1) If Allison vacations in Paris, then she will have to win a scholarship
  - Allison is vacationing in Paris
  - Therefore Allison won a scholarship

$$\begin{array}{rl} & p \rightarrow q \\ & p \\ \hline \therefore & q \end{array}$$

Law of the Syllogism

- Example 2.23: $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$
- Tabular form:

$$\begin{array}{rl} & p \rightarrow q \\ & q \rightarrow r \\ \hline \therefore & p \rightarrow r \end{array}$$

- Example:
  - p1) If 35244 is divisible by 396, then 35244 is divisible by 66
  - p2) If 35244 is divisible by 66, then 35244 is divisible by 3
  - c) Therefore, if 35244 is divisible by 396, then 35244 is divisible by 3

Common Inference Rules

Table 2.19

Rule of Inference	Related Logical Implication	Name of Rule
1) p; $p \to q$; $\therefore q$	$[p \land (p \to q)] \to q$	Rule of Detachment (Modus Ponens)
2) $p \to q$; $q \to r$; $\therefore p \to r$	$[(p \to q) \land (q \to r)] \to (p \to r)$	Law of the Syllogism
3) $p \to q$; $\neg q$; $\therefore \neg p$	$[(p \to q) \land \neg q] \to \neg p$	Modus Tollens
4) p; q; $\therefore p \land q$		Rule of Conjunction
5) $p \lor q$; $\neg p$; $\therefore q$	$[(p \lor q) \land \neg p] \to q$	Rule of Disjunctive Syllogism
6) $\neg p \to F_0$; $\therefore p$	$(\neg p \to F_0) \to p$	Rule of Contradiction

Common Inference Rules (cont.)

7)	$\dfrac{p \land q}{\therefore p}$	$(p \land q) \to p$	Rule of Conjunctive Simplification
8)	$\dfrac{p}{\therefore p \lor q}$	$p \to p \lor q$	Rule of Disjunctive Amplification
9)	$\dfrac{\begin{array}{l} p \land q \\ p \to (q \to r) \end{array}}{\therefore r}$	$[(p \land q) \land [p \to (q \to r)]] \to r$	Rule of Conditional Proof
10)	$\dfrac{\begin{array}{l} p \to r \\ q \to r \end{array}}{\therefore (p \lor q) \to r}$	$[(p \to r) \land (q \to r)] \to [(p \lor q) \to r]$	Rule for Proof by Cases
11)	$\dfrac{\begin{array}{l} p \to q \\ r \to s \\ p \lor r \end{array}}{\therefore q \lor s}$	$[(p \to q) \land (r \to s) \land (p \lor r)] \to (q \lor s)$	Rule of the Constructive Dilemma
12)	$\dfrac{\begin{array}{l} p \to q \\ r \to s \\ \neg q \lor \neg s \end{array}}{\therefore \neg p \lor \neg r}$	$[(p \to q) \land (r \to s) \land (\neg q \lor \neg s)] \to (\neg p \lor \neg r)$	Rule of the Destructive Dilemma

Example 2.31

- If the band could not play rock music or the refreshments were not delivered on time, then the party would have been canceled, and Alicia would have been angry. If the party were canceled, then refunds would have had to be made. No refunds were made
- First, we define the primitives
  - p: The band could play rock music
  - q: The refreshments were delivered on time
  - r: The party was canceled
  - s: Alicia was angry
  - t: Refunds had to be made

Example 2.31 (cont.)

- The argument becomes

$$\begin{array}{ll} & (\neg p \vee \neg q) \rightarrow (r \wedge s) \\ & r \rightarrow t \\ & \neg t \\ \hline \therefore & p \end{array}$$

- Derivations:
  - 1) $r \rightarrow t$ Premise
  - 2) $\neg t$ Premise
  - 3) $\neg r$ 1) and 2) and Modus Tollens
  - 4) $\neg r \vee \neg s$ 3) and Disjunctive Amplification
  - 5) $\neg(r \wedge s)$ 4) DeMorgan's Laws
  - 6) $(\neg p \vee \neg q) \rightarrow (r \wedge s)$ Premise
  - 7) $\neg(\neg p \vee \neg q)$ 6) and 5) and Modus Tollens
  - 8) $p \wedge q$ 7) DeMorgan, Double Negation
  - 9) $\therefore p$ 8) and Conjunctive Simplification

Example 2.32

- Consider the argument

$$\begin{array}{ll} & \neg p \leftrightarrow q \\ & q \rightarrow r \\ & \neg r \\ \hline \therefore & p \end{array}$$

- Let's prove this argument by contradiction: assuming $\neg p$
  - 1) $\neg p \leftrightarrow q$ Premise
  - 2) $(\neg p \rightarrow q) \wedge (q \rightarrow \neg p)$ (1)
  - 3) $\neg p \rightarrow q$ (2) and Conjunctive Simpli.
  - 4) $q \rightarrow r$ Premise
  - 5) $\neg p \rightarrow r$ (3), (4), and Law of Syllogism
  - 6) $\neg p$ Assumed Premise
  - 7) r (5), (6), and Detachment

Example 2.32 (cont.)

- 8) $\neg r$ Premise
- 9) $r \wedge \neg r$ (7), (8), and Conjunction
- 10) $\therefore p$ (6), (9), and Proof by Contradiction

Example 2.33

- Logical equivalence: $[p \to (q \to r)] \iff [(p \land q) \to r]$

$$\begin{array}{ll} & p_1 \\ & p_2 \\ & \cdots \\ & p_n \\ \hline \therefore & q \to r \end{array} \quad \longrightarrow \quad \begin{array}{ll} & p_1 \\ & p_2 \\ & \cdots \\ & p_n \\ & q \\ \hline \therefore & r \end{array}$$

- A concrete example

$$\begin{array}{ll} & u \to r \\ & (r \land s) \to (p \lor t) \\ & q \to (u \land s) \\ & \neg t \\ \hline \therefore & q \to p \end{array} \quad \longrightarrow \quad \begin{array}{ll} & u \to r \\ & (r \land s) \to (p \lor t) \\ & q \to (u \land s) \\ & \neg t \\ & q \\ \hline \therefore & p \end{array}$$

Invalid Argument

- The following argument is invalid if it's possible for all $p_1, p_2, \ldots, p_n$ to be true, but the q is false

$$\begin{array}{ll} & p_1 \\ & p_2 \\ & \cdots \\ & p_n \\ \hline \therefore & q \end{array}$$

- Example 2.34: Prove the following argument is invalid

$$\begin{array}{ll} & p \\ & p \vee q \\ & q \to (r \to s) \\ & t \to r \\ \hline \therefore & \neg s \to \neg t \end{array}$$

Proof: By counter example, let

$\langle p, q, r, s, t\rangle = \langle 1, 0, 1, 0, 1\rangle$

the four premises are 1, but the conclusion is 0

Outline

2.1 Basic Connectives and Truth Tables

2.2 Logical Equivalence: The Laws of Logic

2.3 Logic Implication: Rules of Inference

2.4 The Use of Quantifiers

2.5 Quantifiers, Definitions, and Proofs of Theorems

Definition 2.5

- A statement is an open statement if
  - it contains one or more variables
  - it is not a statement, but
  - it becomes a statement when the variables in it are replaced by certain allowable choices
- Example: $p(x)$: The number $x+3$ is an even number
- Allowable choices is called universe
  - For example: all integers

Open Statements

- Open statements with multiple variables
  - $q(x,y)$: The numbers $y+2$, $x-y$, and $x+2y$ are even integers
  - by default, variables share the same universe
- Some x, y values lead to true statements while others lead to false statements, e.g.,
  - $p(5)$: The number 8 $(5+3)$ is an even number (true statement)
  - $\neg p(7)$: The number 10 is an even integer (false statement)
  - $q(4, 2)$: The numbers 4, 2, and 8 are even integers (true statement)

Quantifier

- Existential quantifier $\exists x \; p(x)$
  - For some *x*, *p(x)* is true
  - For at least one *x*, *p(x)*
  - There exists an *x*, *p(x)*
- Another existential quantifier $\exists x \exists y \; q(x,y)$ or $\exists x,y \; q(x,y)$
- Universal quantifier $\forall x \; p(x)$
  - For each *x*, *p(x)* is true
  - For any *x*, *p(x)* is true
  - For all *x*, *p(x)* is true
- Another universal quantifier $\forall x \forall y \; q(x,y)$ or $\forall x,y \; q(x,y)$

Quantified Statements

- Open statement *r(x)*: *2x* is an even integer
- Quantified statement $\forall x \in \mathbb{N}\ r(x)$, which is a true statement
  - $\exists x \in \mathbb{N}\ r(x)$ is also true
- We call the variables in open statements as free variables, and the "fixed" variables in quantified statements as bound variables (bound by $\exists, \forall$)
- Can we say $\forall x\ p(x) \Rightarrow \exists x\ p(x)$?
- How about $\exists x\ p(x) \Rightarrow \forall x\ p(x)$?

Summary of Quantifier

Table 2.21

Statement	When Is It True?	When Is It False?
$\exists x\ p(x)$	For some (at least one) a in the universe, $p(a)$ is true.	For every a in the universe, $p(a)$ is false.
$\forall x\ p(x)$	For every replacement a from the universe, $p(a)$ is true.	There is at least one replacement a from the universe for which $p(a)$ is false.
$\exists x\ \neg p(x)$	For at least one choice a in the universe, $p(a)$ is false, so its negation $\neg p(a)$ is true.	For every replacement a in the universe, $p(a)$ is true.
$\forall x\ \neg p(x)$	For every replacement a from the universe, $p(a)$ is false and its negation $\neg p(a)$ is true.	There is at least one replacement a from the universe for which $\neg p(a)$ is false and $p(a)$ is true.

Definition 2.6

- Let $p(x)$, $q(x)$ be open statements with the same universe
- Logical equivalence: $\forall x\ [p(x) \iff q(x)]$, when the biconditional $p(a) \leftrightarrow q(a)$ is true for all a in the universe
- Logical implication: $\forall x\ [p(x) \Rightarrow q(x)]$, when the implication $p(a) \rightarrow q(a)$ is true for each a in the universe
- The definition can be readily extended to open statements with multiple variables

Definition 2.7

- For each quantified statement $\forall x \ [p(x) \rightarrow q(x)]$
  - Its contrapositive is: $\forall x \ [\neg q(x) \rightarrow \neg p(x)]$
  - Its converse is: $\forall x \ [q(x) \rightarrow p(x)]$
  - Its inverse is: $\forall x \ [\neg p(x) \rightarrow \neg q(x)]$

- Example 2.40:
  - A quantified statement is logically equivalent to its contrapositive
  - The converse and inverse are logically equivalent

Logical Equivalences/Implication

Table 2.22 Logical Equivalences and Logical Implications for Quantified Statements in One Variable

For a prescribed universe and any open statements $p(x)$, $q(x)$ in the variable x:

$$\exists x\,[p(x) \wedge q(x)] \Rightarrow [\exists x\; p(x) \wedge \exists x\; q(x)]$$

$$\exists x\,[p(x) \vee q(x)] \Leftrightarrow [\exists x\; p(x) \vee \exists x\; q(x)]$$

$$\forall x\,[p(x) \wedge q(x)] \Leftrightarrow [\forall x\; p(x) \wedge \forall x\; q(x)]$$

$$[\forall x\; p(x) \vee \forall x\; q(x)] \Rightarrow \forall x\,[p(x) \vee q(x)]$$

Negating Statements

Table 2.23 Rules for Negating Statements with One Quantifier

$$\neg[\forall x\ p(x)] \Leftrightarrow \exists x\ \neg p(x)$$

$$\neg[\exists x\ p(x)] \Leftrightarrow \forall x\ \neg p(x)$$

$$\neg[\forall x\ \neg p(x)] \Leftrightarrow \exists x\ \neg\neg p(x) \Leftrightarrow \exists x\ p(x)$$

$$\neg[\exists x\ \neg p(x)] \Leftrightarrow \forall x\ \neg\neg p(x) \Leftrightarrow \forall x\ p(x)$$

Example 2.44

- Over the universe of integers, consider two open statements *p(x)*: *x* is odd, and *q(x)*: x^2-*1* is even
- $\forall x \ [p(x) \rightarrow q(x)]$ is true
- Its negation is

$$\neg[\forall x \ [p(x) \rightarrow q(x)]] \iff \exists x \ [\neg(p(x) \rightarrow q(x))]$$
$$\iff \exists x \ [\neg(\neg p(x) \vee q(x))] \iff \exists x \ [p(x) \wedge \neg q(x)]$$

- In English: There exists an odd integer *x*, and x^2-*1* is odd ← a false statement

Example 2.48

- When a statement consists of existential and universal quantifiers, we read them from left to right

- Let $p(x,y)$ be the open statement: $x + y = 17$
  - The following statement reads: There exists an integer y such that for all integers x, $x + y = 17$

$$\exists y \forall x \; p(x, y)$$

Outline

2.1 Basic Connectives and Truth Tables

2.2 Logical Equivalence: The Laws of Logic

2.3 Logic Implication: Rules of Inference

2.4 The Use of Quantifiers

2.5 Quantifiers, Definitions, and Proofs of Theorems

Theorem, Lemma, Corollary

- Theorem: a true statement of mathematical interests
  - Major results that lead to many consequences
- Lemma: a proven statement used to prove theorems
- Corollary: a direct result from theorems
  - Usually given with simple (or even without) proof

Example 2.52

- Consider 13 integers 2, 4, 6, ……, 24, 26
- Theorem: for any $n \in \{2, 4, 6, \ldots, 26\}$, n can be written as the sum of up to three perfect squares
- Proof: Use method of exhaustion
  - 2=1+1, 4=4, 6=4+1+1, 8=4+4
  - 10=9+1, 12=4+4+4, 14=9+4+1, 16=16, 18=16+1+1
  - 20=16+4, 22=9+9+4, 24=16+4+4, 26=25+1

Rule of Universal Specification

- If an open statement is true for all replacements in a given universe, then the open statement is true for each specific individual member in that universe

- $\forall$ can be interpreted as: (i) for all, and (ii) for each

Rule of Universal Generalization

- If an open statement *p(x)* is proved to be true when *x* is replaced by any arbitrarily chosen element *c* from the universe, then the universally quantified statement $\forall x \; p(x)$ if true

- $\forall$ can be interpreted as: (i) for all, (ii) for each, and (iii) for any

- These two rules extend beyond one variable

Example 2.54

- Let p(x), q(x), and r(x) be open statements over the same universe, prove the following argument is valid

$$\begin{array}{rl} & \forall x\ [p(x) \rightarrow q(x)] \\ & \forall x\ [q(x) \rightarrow r(x)] \\ \hline \therefore & \forall x\ [p(x) \rightarrow r(x)] \end{array}$$

  - 1) $\forall x[p(x) \rightarrow q(x)]$ Premise
  - 2) $p(c) \rightarrow q(c)$ Step (1) and Rule of Universal Specification
  - 3) $\forall x[q(x) \rightarrow r(x)]$ Premise
  - 4) $q(c) \rightarrow r(c)$ Step (3) and Rule of Universal Specification
  - 5) $p(c) \rightarrow r(c)$ Steps (2) and (4) and the Law of Syllogism
  - 6) $\therefore \forall x[p(x) \rightarrow r(x)]$ Step (5) and Rule of Universal Generalization

- *c* is a specific but arbitrarily chosen element

Example 2.55 (An App. of 2.54)

- Let
  - *p(x): 3x-7=20*
  - *q(x): 3x= 27*
  - *r(x): x=9*
- The derivation follows the argument in 2.54
  - If *3x-7=20*, then *3x=27*
  - If *3x=27*, then *x=9*
  - Therefore, if *3x-7=20*, then *x=9*

$$\begin{array}{rl} & \forall x\ [p(x) \rightarrow q(x)] \\ & \forall x\ [q(x) \rightarrow r(x)] \\ \hline \therefore & \forall x\ [p(x) \rightarrow r(x)] \end{array}$$

Paragraph-Style Proofs

- The proofs we have done so far list all the details/ reasons
  - Including the straightforward and trivial ones
- In most occasions, we won't list all the reasons, instead, we describe the key ideas of the proof
- Definition 2.8: We call an integer n even if there exists an r so that $n=2r$. If n is not even, then we call it odd. For an odd n, there exists an s so that $n=2s+1$.

Paragraph-Style Proofs (cont.)

- Theorem 2.2: For all integers k and l, if k and l are both odd, then $k+l$ is even
- Proof:
  - Per definition, we write $k=2a+1$ and $l=2b+1$ for some integers a and b
  - Following the commutative and associative laws and distributive law, we write $k+l=(2a+1)+(2b+1)=2(a+b+1)$
  - Given that a and b are integers, let $c=a+b+1$ is also an integer. Following the definition, we have $k+l=2c$ is even

Example 2.57

- Consider the statement
  - If n is an integer, then $\forall n \; n^2 = n$
- Let's check a few possible n values
  - $1^2 = 1$
  - $0^2 = 0$
- But can we say the statement is true?
- NO, because it's easy to find a counter example, e.g., $2^2 \neq 2$ ← This is a proof of the above statement is false
- The true statement is: $\exists n \; n^2 = n$

Another Theorem

- Theorem 2.3: For two odd integers k and l, their product kl is also odd
- Proof:
  - Per definition, we write $k=2a+1$ and $l=2b+1$, for some integers a and b
  - Use algebra, we have $kl=4ab+2a+2b+1=2(2ab+a+b)+1$
  - Per the definition of odd integers, we know kl is also odd because $2ab+a+b$ is an integer

Three Proofs of a Theorem

- Theorem 2.4: If *m* is an even integer, then *m+7* is odd
  - Direct proof
  - By contradiction: Assume that *m+7* is even.
    - *m+7=2c => m = 2c-7 => m = 2(c-4)+1*
    - *m* is odd?
    - Assume $\neg q(x)$, derive $\neg p(x)$.
  - By contraposition
    - We show if m+7 is even, then m is odd
    - Then because $\forall x\ [p(x) \rightarrow q(x)]$ and $\forall x\ [\neg q(x) \rightarrow \neg p(x)]$ are logically equivalent, the theorem follows
    - Assume $p(x) \wedge \neg q(x)$, derive F_0.
- Try direct proof first, then fall back to indirect methods

When to Use Indirect Proof?

- Theorem 2.5: For all positive real numbers x and y, if the product xy exceeds 25, then $x>5$ or $y>5$
  - Proved by contraposition
- Proof:
  - Consider $0 < x, y \leq 5$, we write $0 = 0 \cdot 0 < xy \leq 5 \cdot 5 = 25$
  - This means xy never exceeds 25
  - Since an implication is logically equivalent to its contrapositive, the above implication yields the proof

Take-home Exercises

- Exercise 2.1: 4, 6, 13, 17
- Exercise 2.2: 6, 14, 15, 19
- Exercise 2.3: 3, 8, 10, 12
- Exercise 2.4: 3, 6, 8, 19
- Exercise 2.5: 7, 19, 21, 24