Department of Computer Science
National Tsing Hua University

CS 2336: Discrete Mathematics

Chapter 8

The Principle of Inclusion and Exclusion

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Outline

8.1 The Principle of Inclusion and Exclusion

8.2 Generalizations of the Principle

8.3 Derangement: Nothing is in Its Right Place

8.4 Rook Polynomials

8.5 Arrangements with Forbidden Positions

Notations

- Let S be a set and $|S|=N$. Let $c_1, c_2, \ldots, c_t$ be a collection of t conditions or properties, each may be satisfied by some elements of S.
- For an $1 <= i <= t$, $N(c_i)$ denotes the number of elements in S that satisfy condition c_i.
- $N(c_i c_j)$ denotes the number of elements in S that satisfy both conditions c_i and c_j, and perhaps others.
- For an $1 <= i <= t$, $N(\bar{c}_i) = N - N(c_i)$ denotes the number of elements not satisfy condition c_i.
- Also define $N(\bar{c}_i \bar{c}_j)$ and $N(\overline{c_i c_j})$.

Principle of Inclusion and Exclusion

- Number of elements of S that satisfy none of the condition $c_i, 1 \leqslant i \leqslant t$, is denoted by $\bar{N} = N(\bar{c_1}\bar{c_2}\cdots\bar{c_t})$

$$\bar{N} = N - [N(c_1) + N(c_2) + \cdots + N(c_t)]$$
$$+ [N(c_1c_2) + N(c_1c_3) + \cdots + N(c_1c_t) + N(c_2c_3) + \cdots + N(c_{t-1}c_t)]$$
$$- [N(c_2c_2c_3) + N(c_1c_2c_4) + \cdots + N(c_1c_2c_t) + N(c_1c_3c_4) + \cdots$$
$$+ N(c_1c_3c_t) + \cdots + N(c_{t-2}c_{t-1}c_t)]]$$
$$+ \cdots + (-1)^t N(c_1c_2c_3\cdots c_t)$$

or

$$\bar{N} = N - \sum_{1\leqslant i\leqslant t} N(c_i) + \sum_{1\leqslant i,j\leqslant t} N(c_ic_j) - \sum_{1\leqslant i,j,k\leqslant t} N(c_ic_jc_k) + \cdots$$
$$+ (-1)^t N(c_1c_2c_3\cdots c_t)$$

Proof by Counting

$$\bar{N} = N - \sum_{1 \leqslant i \leqslant t} N(c_i) + \sum_{1 \leqslant i,j \leqslant t} N(c_i c_j) - \sum_{1 \leqslant i,j,k \leqslant t} N(c_i c_j c_k) + \cdots + (-1)^t N(c_1 c_2 c_3 \cdots c_t)$$

- Goal: show any $x \in S$ contributes the same amount (either *0* or *1*) to LHS and RHS
- Case I: What if *x* satisfies none of the conditions?
- Case II: If *x* satisfies exactly *r* conditions. LHS is always 0. RHS is
  - (1) once in N (first term)
  - (2) *r* times in the second terms

Proof by Counting (cont.)

$$\bar{N} = N - \sum_{1 \leqslant i \leqslant t} N(c_i) + \sum_{1 \leqslant i,j \leqslant t} N(c_i c_j) - \sum_{1 \leqslant i,j,k \leqslant t} N(c_i c_j c_k) + \cdots$$
$$+ (-1)^t N(c_1 c_2 c_3 \cdots c_t)$$

- Case II: If x satisfies exactly r conditions. RHS is
  - (3) $\binom{r}{2}$ times in the third terms
  - (4) $\binom{r}{3}$ times in the fourth terms
  - ....
  - (Last) $\binom{r}{r}$ time in the last term.
- Hence, the RHS is

$$1 - r + \binom{r}{2} - \binom{r}{3} + \cdots + (-1)^r \binom{r}{r} = [1 + (-1)]^r = 0^r = 0$$

  - Per Corollary 1.1

More Notations

- The number of elements in S that satisfy at least one condition c_i, is given by $N(c_1 \text{ or } c_2 \text{ or } \cdots \text{ or } c_t) = N - \bar{N}$
- For brevity, we write

$$S_0 = N$$
$$S_1 = [N(c_1) + N(c_2) + \cdots N(c_t)]$$
$$\cdots\cdots$$
$$S_k = \sum N(c_{i_1}, c_{i_2}, \cdots c_{i_k}),\ 1 \leqslant k \leqslant t$$

, goes over all size k collection from t conditions, hence it has $\binom{t}{k}$ entries

- Hence we have $\bar{N} = S_0 - S_1 + S_2 - S_3 + \cdots + (-1)^t S_t$

Simple Example 1

- Ex 8.4: Find the number of positive integer between *1* and *100* (inclusive), where n is not divisible by *2*, *3*, or *5*
  - c_1: if n is divisible by *2*
  - c_2: if n is divisible by *3*
  - c_3: if n is divisible by *5*
- What is $N(\bar{c_1}\bar{c_2}\bar{c_3})$?

$$N(c_1) = \lfloor 100/2 \rfloor$$

$$N(c_1 c_2) = \lfloor 100/(2 \times 3) \rfloor$$

$\dots\dots\dots$

Simple Example 2

- Ex 8.7: In how many ways can *26* letters be permuted so that none of the patterns: car, dog, pun, or byte occurs?
  - Let S be the set of all permutations of letters, $|S| = 26!$
  - c_i is the number of permutation contains the i-th pattern
- $N(c_1) = 24!, N(c_2) = N(c_3) = 24!, N(c_4) = 23!$
- $N(c_1 c_3) = N(c_2 c_3) = 22! \quad N(c_1 c_4) = N(c_2 c_4) = N(c_3 c_4) = 21!$
- ....
- $N(\bar{c_1}\bar{c_2}\bar{c_3}\bar{c_4}) = 26! - [3(24!) + 23!] + [3(22!) + 3(21!)] - [20! + 3(19!)] + 17!$

Simple Example 3

- An engineer is building two-way roads to connect five villages, s.t. no village will be isolated. In how many ways can he do this? No-loop is considered.
- Let S be all loop-free undirected graphs G on $V=\{a,b,c,d,e\}$. $S_0=2^{10}$, because there are 10 possible two-way roads for five villages.

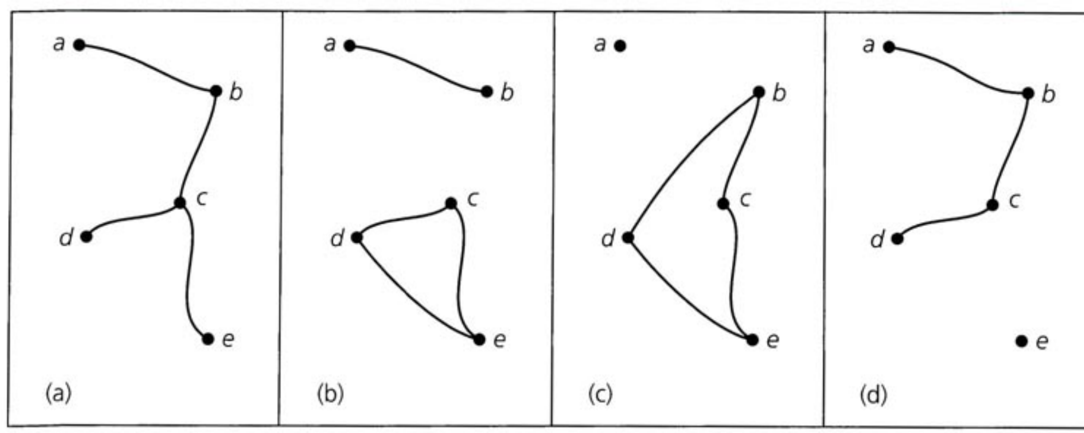

Figure 8.3

Simple Example 3 (cont.)

- Let c_i be the condition that the i-th village is isolated. The answer to our problem is $N(\bar{c_1}\bar{c_2}\bar{c_3}\bar{c_4}\bar{c_5})$
- $N(c_i)=2^6$, why?
- $N(c_ic_j)=2^3$, why?
- Ans: $2^{10} - \binom{5}{1}2^6 + \binom{5}{2}2^3 - \binom{5}{3}2^1 + \binom{5}{4}2^0 - \binom{5}{5}2^0 = 768$

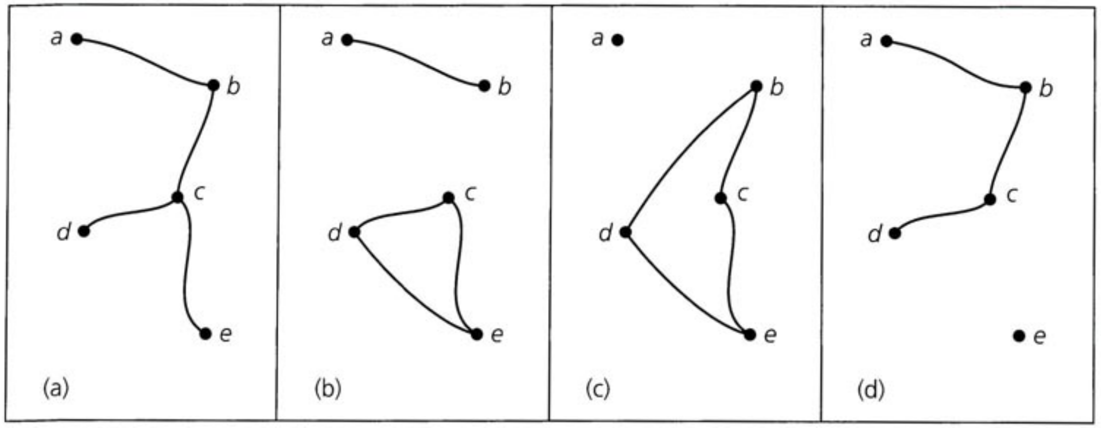

Figure 8.3

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8.2 Generalizations of the Principle

8.3 Derangement: Nothing is in Its Right Place

8.4 Rook Polynomials

8.5 Arrangements with Forbidden Positions

Notations

- Let S be a set and $|S|=N$. Let $c_1, c_2, \ldots, c_t$ be a collection of t conditions or properties, each may be satisfied by some elements of S.
- Let E_m be the number of elements in S that satisfy exactly m of the t conditions
  - We knew how to compute E_0
- $E_1 = N(c_1\bar{c_2}\cdots\bar{c_t}) + N(\bar{c_1}c_2\cdots\bar{c_t}) + \cdots + N(\bar{c_1}\bar{c_2}\cdots c_t)$
- $E_2 = N(c_1c_2\bar{c_3}\cdots\bar{c_t}) + N(c_1\bar{c_2}c_3\cdots\bar{c_t}) + \cdots + N(\bar{c_1}\bar{c_2}\bar{c_3}\cdots c_{t-1}c_t)$

Generalized Formula

- E_m is given by:

$$E_m = S_m - \binom{m+1}{1} S_{m+1} + \binom{m+2}{2} S_{m+2} - \cdots + (-1)^{t-m} \binom{t}{t-m} S_t$$

- Proof Sketch:

- Case I: x satisfies fewer than m conditions. Contributes *0* to both LHS and RHS.

- Case II: x satisfies exactly m conditions. It is counted once in E_m, and once in S_m, but not in S_{m+1}, …, S_i.

Generalized Formula (cont.)

$$E_m = S_m - \binom{m+1}{1} S_{m+1} + \binom{m+2}{2} S_{m+2} - \cdots + (-1)^{t-m} \binom{t}{t-m} S_t$$

- Case III: x satisfies r conditions, where $m < r \leqslant t$. x contributes nothing in LHS. It is counted $\binom{r}{m}$ times in S_m, $\binom{r}{m+1}$ times in S_{m+1}, …, and $\binom{r}{r}$ in S_r, but *0* time for anything beyond r. Hence we have the count:

$$\binom{r}{m} - \binom{m+1}{1}\binom{r}{m+1} + \binom{m+2}{2}\binom{r}{m+2} + \cdots + (-1)^{r-m}\binom{r}{r-m}\binom{r}{r}$$

  - some derivations lead to *0* in RHS.

A Simple Example

- $E_1 = N(c_1) + N(c_2) + N(c_3) - 2[N(c_1c_2) + N(c_1c_3) + N(c_2c_3)] + 3N(c_1c_2c_3)$
- $E_2 = N(c_1c_2) + N(c_1c_3) + N(c_2c_3) - 3N(c_1c_2c_3)$
- $E_3 = N(c_1c_2c_3)$

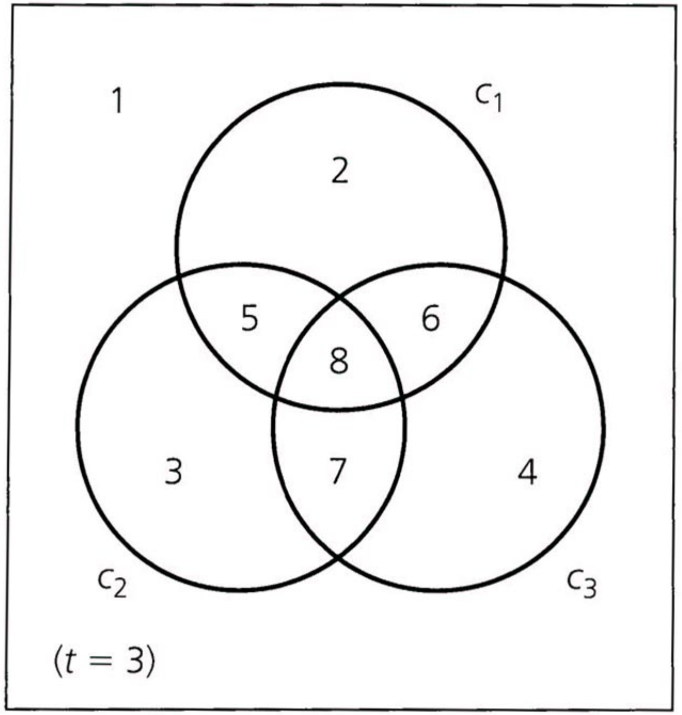

Figure 8.4

Another Generalization

- Let L_m denotes the number of elements of S that satisfy at least m of the t conditions.

$$L_m = S_m - \binom{m}{m-1} S_{m+1} + \binom{m+1}{m-1} S_{m+2} - \cdots + (-1)^{t-m} \binom{t-1}{m-1} S_t$$

- What is L_2 in the figure?

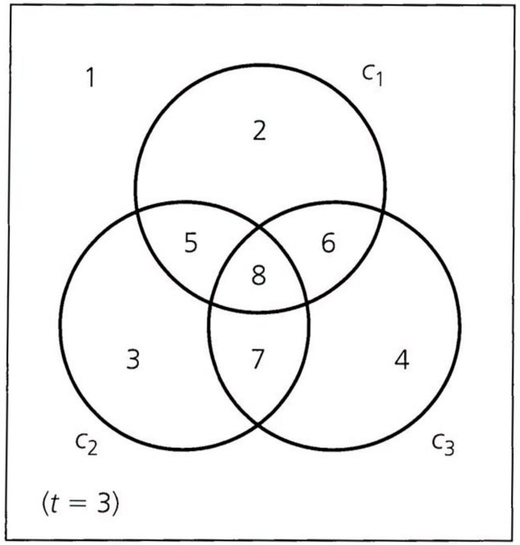

Figure 8.4

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Derangements

- Ex 8.12: Ralph bets on ten chosen horses in a race. In how many ways can they reach the finish line so that he loses all the bets?
- Equivalent to: In how many ways we can arrange *1*, *2*, …, *10* so that *1* is not in first place, *2* is not in second plan, …., and *10* is not in tenth place?
- These arrangements are called derangements of *1*, *2*, *3*, …, *10*

Derangements (cont.)

- An arrangement is said to satisfy condition c_i, if integer i is in the i-th place
- The number of derangements d_{10} is given by

$$d_{10} = N(\bar{c_1}\bar{c_2}\cdots\bar{c_{10}}) = 10! - \binom{10}{1}9! + \binom{10}{2}8! - \binom{10}{3}7! + \cdots + \binom{10}{10}0!$$

$$= 10![1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{1}{10!}] \approx (10!)e^{-1}$$

- The probability that Ralph will lose every bet is about $\frac{10!e^{-1}}{10!} = e^{-1}$
  - Not a bad approximation with number of horses is 11, 12, 13, …..

Maclaurin Series for Exp Function

- We knew $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
- Hence, $e^{-1} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots$
- With $k \geqslant 7$, $\sum_{n=0}^{k} \frac{(-1)^n}{n!}$ can approximate e^{-1} well

Two Simple Examples

- Ex 8.13: The number of derangements of *1*, *2*, *3*, *4*

$$d_4 = N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4) = 4! - \binom{4}{1}3! + \binom{4}{2}2! - \binom{4}{3}1! + \binom{4}{4}0!$$

$$= 4!(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}) = 12 - 4 + 1 = 9$$

- Ex 8.14: Peggy assigns 7 books to 7 reviewers: one book for each reviewer in the first week, and another book for each reviewer in the second week. How many ways can she distribute the books so that she gets two reviewers of each book?
  - 7! for the first week, so $7!d_7 \approx 7!^2e^{-1}$ in total

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Chessboard and Rook

- Shaded squares are not part of a chessboard C
- Rook (or castle) can move horizontally or vertically over all unoccupied spaces in each turn
  - Ex: Where can a rook move if it's at 3, or 5?

- $r_k(C)$: the number of ways, k rooks can be placed on the unshaded squares so that no two of them can take each other.

Figure 8.6

Rook Polynomial

- $r_k(C)$: the number of ways, k rooks can be placed on the unshaded squares so that no two of them can take each other.
- Rook Polynomial: $r(C,x) = \sum_{\forall k} r_k(C)x^k$
- Ex: $r_0(C)=1$, $r_1(C)=6$, $r_2(C)=8$, $r_3(C)=2$
  - Thus $r(C,x) = 1 + 6x + 8x^2 + 2x^3$

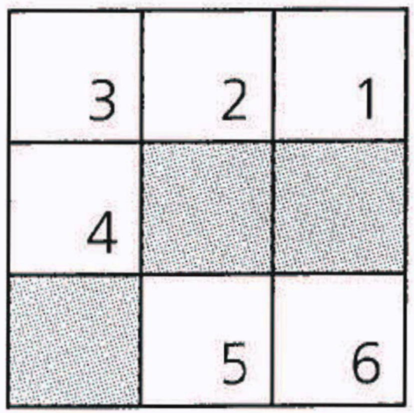


Figure 8.6

Subboards

- The above approach is tedious, and we want to break a chessboard into multiple subboards
- 2 subboards: C_1 (upper-left) and C_2 (lower-right)
- The are disjoint, and we have

$$r(C_1, x) = 1 + 4x + 2x^2$$

$$r(C_2, x) = 1 + 7x + 10x^2 + 2x^3$$

$$r(C, x) = 1 + 11x + 40x^2 + 56x^3 + 28x^4 + 4x^5$$

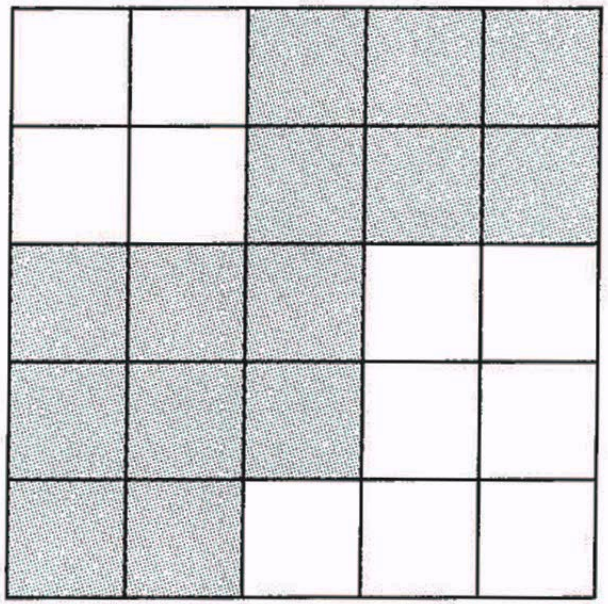


Figure 8.7

Subboards (cont.)

$$r(C_1, x) = 1 + 4x + 2x^2$$

$$r(C_2, x) = 1 + 7x + 10x^2 + 2x^3$$

$$r(C, x) = 1 + 11x + 40x^2 + 56x^3 + 28x^4 + 4x^5$$

- Observe $r(C, x) = r(C_1, x)r(C_2, x)$
- Why? Consider $r_3(C)$
  - 3 are in C_2
  - 2 in C_2 and 1 in C_1
  - 2 in C_1 and 1 in C_2

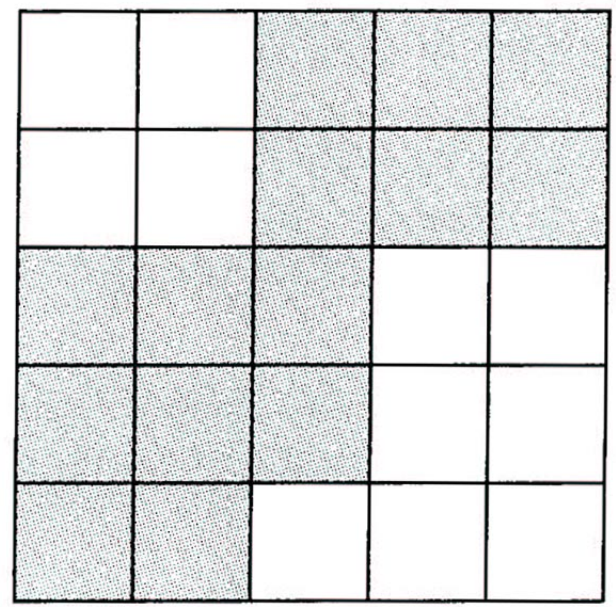


Figure 8.7

Subboards (cont.)

- Generalized: If C is a chessboard consisting of pairwise disjoint subboards $C_1, C_2, \ldots, C_n$, then

$$r(C, x) = r(C_1, x)r(C_2, x) \cdots r(C_n, x)$$

Breaking Chessboards

- What if subboards are not disjoint?
- For a square of C, consider two cases
  - Place a rook on it
  - Do not place a rook on it
- Then, we have $r_k(C) = r_{k-1}(C_s) + r_k(C_e)$, which allows us to work on smaller chessboards

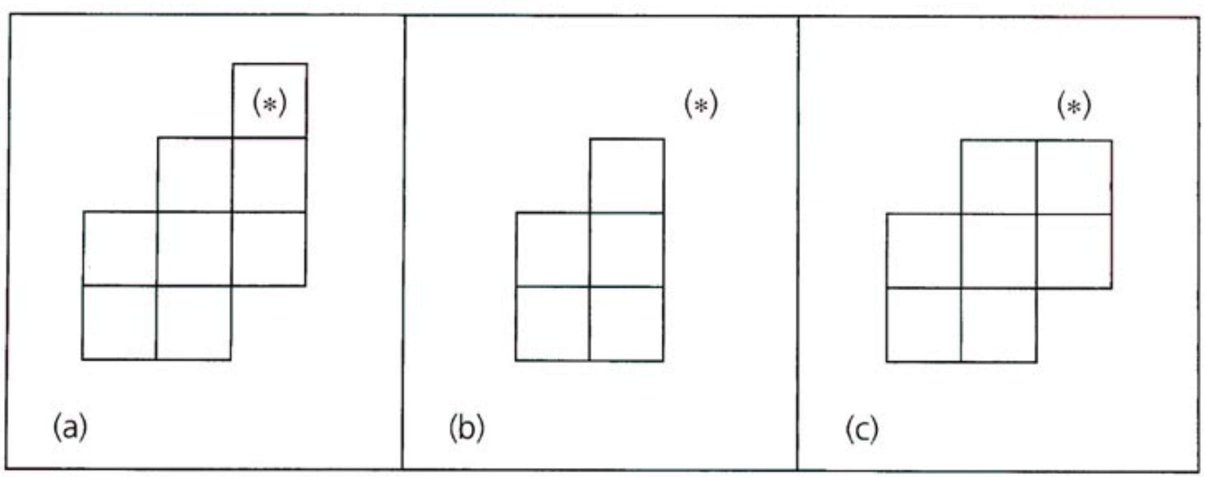
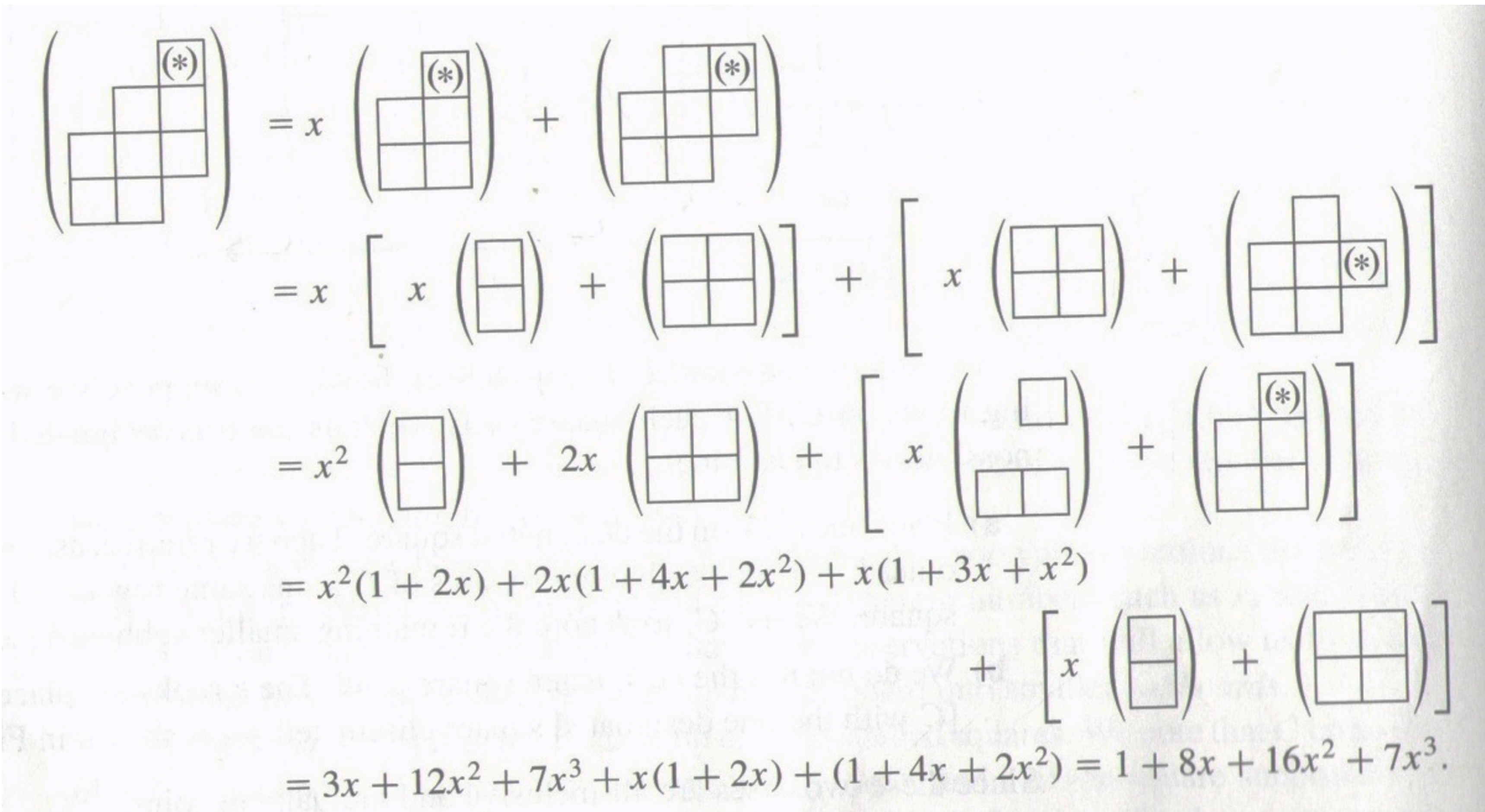


Figure 8.8

Breaking Chessboards (cont.)

- From $r_k(C) = r_{k-1}(C_s) + r_k(C_e)$, we derive

$$r(C, x) = xr(C_s, x) + r(C_e, x)$$



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Application of Rook Polynomial

- Ex 8.15: Four relatives R_1, R_2, R_3, R_4, who hate each other, and can sit at Table T_1, T_2, T_3, T_4, but with the following constraints
  - R_1 will not sit at T_1 or T_2, R_2 will not sit at T_2, R_3 will not sit at T_3 or T_4, R_4 will not sit at T_4 or T_5
- c_i be the condition, R_i is in a forbidden (shaded) position.
  - $N(c_1)=N(c_3)=N(c_4)=4!+4!$, why?
  - $N(c_2)=4!$
  - $N(c_1c_2)=3!$, $N(c_1c_3)=4(3!)$, why?

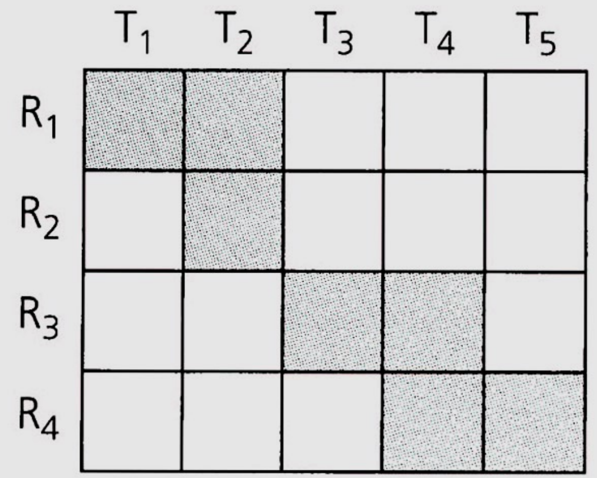

Figure 8.9

Application of Rook Polynomial (cont.)

- If we continue, we have

$$S_1 = 7(4!) = 7(5-1)!,\ S_2 = 16(3!) = 16(5-2)!$$

- More general, $S_i = r_i(5-i)!,\ \forall 0 \leqslant i \leqslant 4$, where r_i is the number of ways to place nontaking rooks on the shaded squares
- This allows us to use $r(C,x)$,

  to derive $N(\bar{c_1}\bar{c_2}\bar{c_3}\bar{c_4})$

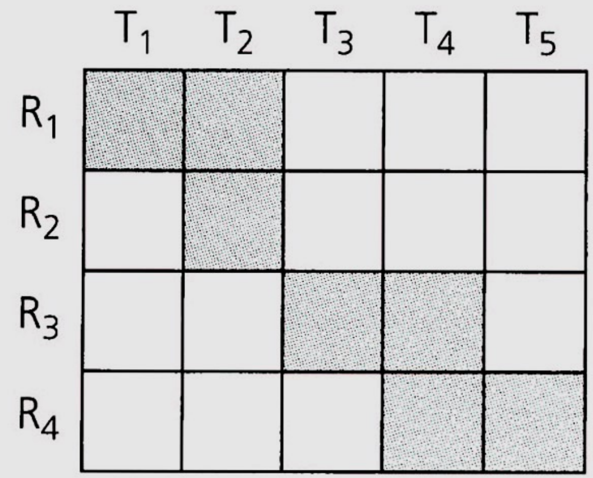

Figure 8.9

Application of Rook Polynomial (cont.)

- Specifically, using disjoint subboards, we get

$$r(C, x) = 1 + 7x + 16x^2 + 13x^3 + 3x^4$$

- Which leads to $N(\bar{c_1}\bar{c_2}\bar{c_3}\bar{c_4}) = S_0 - S_1 + S_2 - S_3 + S_4$

$$= 5! - 7(4!) + 16(3!) - 13(2!) + 3(1!) = \sum_{i=0}^{4} (-1)^i r_i (5-i)! = 25$$

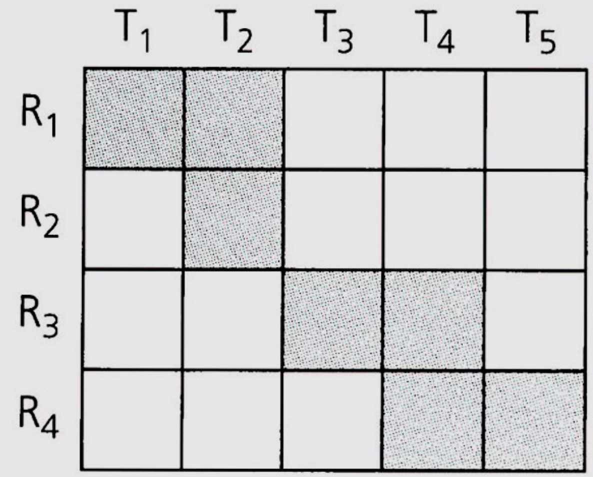

Figure 8.9

Renumbering May Help Calculations

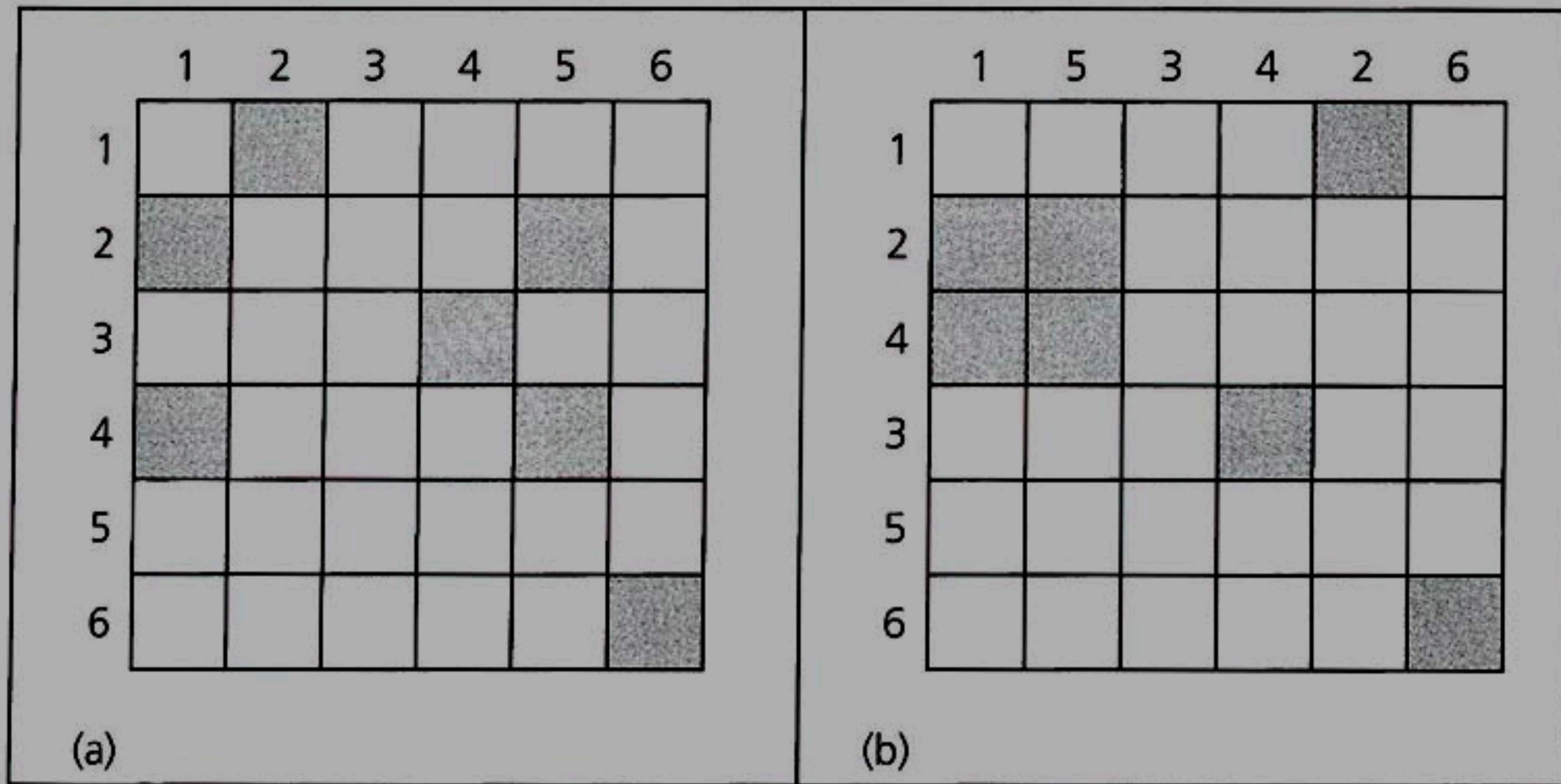

Figure 8.10

Counting One-to-One Functions

- $A=\{1,2,3,4\}$ and $B=\{u,v,w,x,y,z\}$. How many 1-1 functions from A to B satisfy none of the following conditions?
  - c_1: $f(1)=u$ or v, c_2: $f(2)=w$, c_3: $f(3)=w$ or x, and c_4: $f(4)=x,y,$ or z
- We are interested in the shaded area

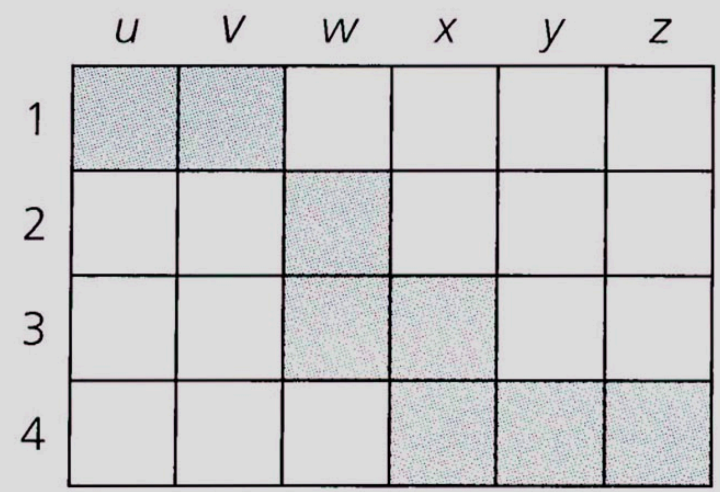

Figure 8.11

Counting One-to-One Functions (cont.)

- We have

$$r(C,x) = (1+2x)(1+6x+9x^2+2x^3) = 1 + 8x + 21x^2 + 20x^3 + 4x^4$$

- Then,

$$N(\bar{c_1}\bar{c_2}\bar{c_3}\bar{c_4}) = S_0 - S_1 + S_2 - S_3 + S_4 = \sum_{i=0}^{4}(-1)^i r_i \frac{(6-i)!}{2!} = 76$$

- There are 76 1-1 functions with none of the conditions satisfied

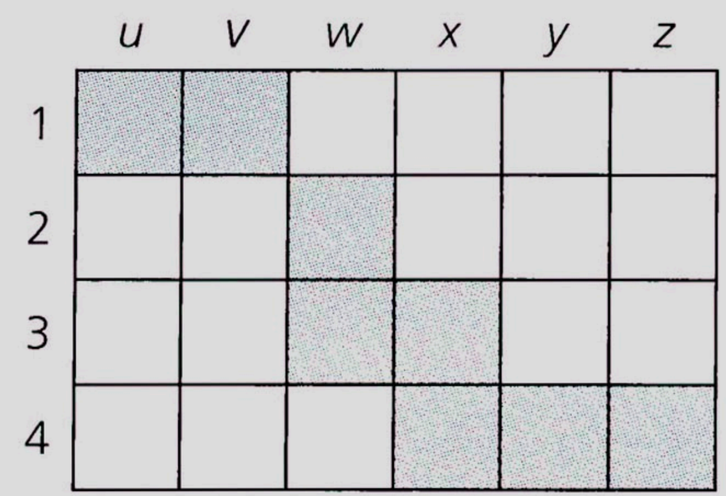

Figure 8.11

Take-home Exercises

- Exercise 8.1: 1, 6, 8, 16, 20
- Exercise 8.2: 2, 3, 8
- Exercise 8.3: 1, 4, 6, 9, 10
- Exercise 8.4 and 8.5: 4, 5, 7, 8, 12