Department of Computer Science
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CS 2336: Discrete Mathematics

Chapter 9

Generating Functions

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Outline

9.1 Introductory Examples

9.2 Definition and Examples: Calculation Techniques

9.3 Partitions of Integers

9.4 The Exponential Generating Function

9.5 The Summation Operator

Motivate Example

- Ex 9.1: How many ways to distribute *12* oranges to three kids: G, M, and F, so that G gets at least *4*, M and F get at least *2*, but F gets no more than *5*?
  - Not hard to see, we are looking for no. solutions of $c_1+c_2+c_3=12$, where $4 <= c_1$, $2<=c_2$, and $2<=c_3<=5$.

Table 9.1

G	M	F	G	M	F
4	3	5	6	2	4
4	4	4	6	3	3
4	5	3	6	4	2
4	6	2	7	2	3
5	2	5	7	3	2
5	3	4	8	2	2
5	4	3			
5	5	2			

Motivate Example (cont.)

- Consider two example: 4+3+5=12 and 4+4+4=12
- They are correspondent to $x^4x^3x^5$ and $x^4x^4x^4$ in the following polynomial multiplication
  - $(x^4 + x^5 + x^6 + x^7 + x^8)(x^2 + x^2 + x^4 + x^5 + x^6)(x^2 + x^3 + x^4 + x^5)$
- So, the no. ways for distribution is the coefficient of x^{12}
- $f(x) = (x^4 + x^5 + x^6 + x^7 + x^8)(x^2 + x^2 + x^4 + x^5 + x^6)(x^2 + x^3 + x^4 + x^5)$ is called the generating function for the distribution.

Another Example

- Ex 9.2: Assuming there are unlimited number of red, green, white, and black beans. How many ways we can choose 24 beans, so that we have even number of white beans and at least six black beans?
  - Red/Green: $1 + x + x^2 + \cdots + x^{24}$
  - White: $1 + x^2 + x^4 + x^6 + \cdots + x^{24}$
  - Black: $x^6 + x^7 + x^8 + \cdots + x^{24}$
- What is the generation function *f(x)*?
- How to use the generation function?

Last Example

- Ex 9.3: How many nonnegative integer solutions for $c_1+c_2+c_3+c_4=25$
  - Polynomial: $f(x) = (1 + x + x^2 + \cdots + x^{25})^4$
  - Power series: $g(x) = (1 + x + x^2 + \cdots + x^{25} + x^{26} + \cdots)^4$
- They are the same
  - Same as no. ways to distribute 10,000 coins among 4 kids so that they got 25 coins in total
- Why bother to deal with power series? Sometimes they are easier to handle than polynomial

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Power Series

- Infinite series in this form:

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c)^1 + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

- Taylor series: $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n =$

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots.$$

  - When *a=0*, we call it Maclaurin series

Generating Function

- Definition: Let a_0, a_1, …., be a sequence of real number. The function

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{i=0}^{\infty} a_i x^i$$

is the generating function for the given sequence.

- Ex 9.4: We know $(1+x)^n = \binom{n}{0} + \binom{n}{1} x + \cdots + \binom{n}{n} x^n$
  - What's the sequence for the generating function $(1+x)^n$?

Examples

- Ex 9.5 (a): From $(1 - x^{n+1}) = (1 - x)(1 + x + x^2 + \cdots + x^n)$ we have $\frac{1 - x^{n+1}}{1 - x} = 1 + x + x^2 + \cdots + x^n$
  - What is the generating function and sequence?
- Ex 9.5 (b): What can we say about $1 = (1 - x)(1 + x + x^2 + \cdots)$
- Ex 9.5 (c): Take derivative, we have

$$\frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x)^2} = \frac{d}{dx}(1 + x + x^2 + \cdots) = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

  - $\frac{1}{(1-x)^2}$ is the generating function of which sequence?
  - How about $\frac{x}{(1-x)^2}$?

More Examples

- Ex 9.6 (a): We have $\frac{1}{1-y} = 1 + y + y^2 + y^3 + \cdots$
  - Substitute *2x* for *y*, what are the generating function and sequence?
- Ex 9.6 (b): We know the generating function of f(x)=1/(1-x) is 1, 1, 1, …..
  - What is the generating function of 1, 1, 0, 1, 1, …. ?
- Ex 9.6 (c): Find the generating function for the sequence 0, 2, 6, 12, 20, 32, 42, … ? ← $a_n=n^2+n$
  - From 9.5, we know $\frac{x(x+1)}{(1-x)^3}$ is the generating function of n^2, and $\frac{x}{(1-x)^2}$ is that of *n*

Extending Binomial Theorem

- For positive integer *n*, we have

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \cdots \binom{n}{n}x^n$$

- What if *n<0*, or non-integer *n*?
  - For $n \in \mathbb{R}$, we define $\binom{n}{r} = \frac{n(n-1)\cdots(n-r+1)}{r!}$
  - For $n \in \mathbb{Z}^+$, we define

$$\binom{-n}{r} = \frac{(-n)(-n-1)\cdots(-n-r+1)}{r!} =$$

$$(-1)^r \frac{(n+r-1)!}{(n-1)!r!} = (-1)^r \binom{n+r-1}{r}$$

- Last, for any real number *n*, we have $\binom{n}{0} = 1$

Example

- Ex 9.7: With Maclaurin series expansion, we write

$$(1+x)^{-n} = \sum_{r=0}^{\infty} (-1)^r \binom{n+r-1}{r} x^r$$

  - Hence $(1+x)^{-n} = \binom{-n}{0} + \binom{-n}{1} x + \cdots = \sum_{r=0}^{\infty} \binom{-n}{r} x^r$

- Ex 9.8: Find the coefficient of x^5 in $(1-2x)^{-7}$
  - We have $(1-2x)^{-7} = (1+y)^{-7} = \sum_{r=0}^{\infty} \binom{-7}{r} y^r = \sum_{r=0}^{\infty} \binom{-7}{r} (-2x)^r$
  - So the coefficient of x^5 is

$$\binom{-7}{5}(-2)^5 = (-1)^5 \binom{7+5-1}{5}(-32) = 32\binom{11}{5} = 14784$$

More Identities

For all $m, n \in \mathbf{Z}^+$, $a \in \mathbf{R}$,

1) $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$

2) $(1+ax)^n = \binom{n}{0} + \binom{n}{1}ax + \binom{n}{2}a^2x^2 + \cdots + \binom{n}{n}a^nx^n$

3) $(1+x^m)^n = \binom{n}{0} + \binom{n}{1}x^m + \binom{n}{2}x^{2m} + \cdots + \binom{n}{n}x^{nm}$

4) $(1-x^{n+1})/(1-x) = 1 + x + x^2 + \cdots + x^n$

5) $1/(1-x) = 1 + x + x^2 + x^3 + \cdots = \sum_{i=0}^{\infty} x^i$

6) $1/(1-ax) = 1 + (ax) + (ax)^2 + (ax)^3 + \cdots$

$= \sum_{i=0}^{\infty}(ax)^i = \sum_{i=0}^{\infty} a^i x^i$

$= 1 + ax + a^2x^2 + a^3x^3 + \cdots$

7) $1/(1+x)^n = \binom{-n}{0} + \binom{-n}{1}x + \binom{-n}{2}x^2 + \cdots$

$= \sum_{i=0}^{\infty} \binom{-n}{i}x^i$

$= 1 + (-1)\binom{n+1-1}{1}x + (-1)^2\binom{n+2-1}{2}x^2 + \cdots$

$= \sum_{i=0}^{\infty}(-1)^i\binom{n+i-1}{i}x^i$

8) $1/(1-x)^n = \binom{-n}{0} + \binom{-n}{1}(-x) + \binom{-n}{2}(-x)^2 + \cdots$

$= \sum_{i=0}^{\infty} \binom{-n}{i}(-x)^i$

$= 1 + (-1)\binom{n+1-1}{1}(-x) + (-1)^2\binom{n+2-1}{2}(-x)^2 + \cdots$

$= \sum_{i=0}^{\infty} \binom{n+i-1}{i}x^i$

If $f(x) = \sum_{i=0}^{\infty} a_i x^i$, $g(x) = \sum_{i=0}^{\infty} b_i x^i$, and $h(x) = f(x)g(x)$, then $h(x) = \sum_{i=0}^{\infty} c_i x^i$, where for all $k \geq 0$,

$$c_k = a_0b_k + a_1b_{k-1} + \cdots + a_{k-1}b_1 + a_kb_0 = \sum_{j=0}^{k} a_j b_{k-j}.$$

Some Applications

- Ex 9.11: In how many ways can we select, with repetitions allows, *r* objects from *n* distinct objects?
  - Generation function: $f(x) = (1 + x + x^2 + x^3 + \cdots)^n$
  - With identity 5: $f(x) = (\frac{1}{1-x})^n$
  - With identity 8: $f(x) = (\frac{1}{1-x})^n = \sum_{i=0}^{\infty} \binom{n+i-1}{i} x^i$
  - What's the coefficient of x^r?

More Examples

- Ex 9.14: How many ways can we distribute 24 rifle shells to 4 police officers, so that each of them gets at least 3, but no more than 8 shells?
  - The generating function is $f(x) = (x^3 + x^4 + \cdots + x^8)^4$
  - We look for the coefficient of x^{24}
  - $f(x) = x^{12}(1 + x + x^2 + \cdots + x^5)^4 = x^{12}(\frac{1 - x^6}{1 - x})^4$
  - We actually are looking for the coefficient of x^{12} in $(\frac{1 - x^6}{1 - x})^4$
  - Which is $[\binom{-4}{12}(-1)^{12} - \binom{4}{1}\binom{-4}{6}(-1)^6 + \binom{4}{2}\binom{-4}{0}] = 125$

Convolution of Sequences

- Ex 9.19: Consider two generating functions
  - $f(x) = \frac{x}{(1-x)^2}$ generates $a_k=k$
  - $g(x) = \frac{x(x+1)}{(1-x)^3}$ generates $b_k=k^2$
  - $h(x) = f(x)g(x)$ generates c_k, where $c_k = \sum_{i=0}^{k} i(k-i)^2$

- The sequence c_0, c_1, c_2, …. is called the convolution of sequences a_0, a_1, a_2, … and b_0, b_1, b_2, …

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Partitioning Integers

- Let $p(n)$ be the number of ways to partition n into positive summands, without regard to order.
  - p(1)=1: 1
  - p(2)=2: 2=1+1
  - p(3)=3: 3=2+1=1+1+1
  - p(4)= 5: 4=3+1=2+2=2+1+1=1+1+1+1
  - ….
- Idea: we need to keep track of numbers of 1, 2, and so on.
  - But how?

Generating Function

- For p(6), the generating function is

$$f(x) = (1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)\cdots(1 + x^6 + x^{12} + \cdots)$$

or

$$g(x) = (1 + x + x^2 + \cdots + x^6)(1 + x^2 + x^4 + \cdots + x^6)\cdots(1 + x^6)$$

- In fact: $f(x) = \frac{1}{(1-x)}\frac{1}{(1-x^2)}\cdots\frac{1}{(1-x^6)} = \prod_{i=1}^{6}\frac{1}{(1-x^i)}$

Examples

- Ex 9.21: Find the number of ways to buy advertisement minutes *n* (an integer) if time slots are sold as blocks of 30, 60, or 120 second.
  - Use 30 sec as the time unit, we look for no. solutions for $a+2b+4c=2n$, where $a, b, c >= 0$
  - Generating function is

$$f(x) = (1+x+x^2+\cdots)(1+x^2+x^4+\cdots)(1+x^4+x^8+\cdots) = \frac{1}{1-x}\frac{1}{1-x^2}\frac{1}{1-x^4}$$

  - The answer for the number of partitions of $2n$ is the coefficient of x^{2n}

Distinct Summands

- Ex 9.22: Let $p_d(n)$ be the number of partitions of a positive integer n into distinct positive summands
  - So each integer either appears once or does not appear
  - Generating function is
$$p_d(x) = (1+x)(1+x^2)(1+x^3)\cdots = \prod_{i=1}^{\infty}(1+x^i)$$
  - $p_d(n)$ is the coefficient of x^n in $(1+x)(1+x^2)\ldots(1+x^n)$
  - Example: $p_d(6) = 4$: 6=1+5=2+4=1+2+3

Odd Summands

- Ex 9.23: Let $p_o(n)$ be the number of partitions of a positive integer n into positive odd summands
  - Generating function is

$$p_o(x) = (1+x+x^2+\cdots)(1+x^3+x^6+\cdots)(1+x^5+x^{10}+\cdots)\cdots = \frac{1}{1-x}\frac{1}{1-x^3}\frac{1}{1-x^5}\cdots$$

  - Note that: $1+x = \frac{1-x^2}{1-x},\ 1+x^2 = \frac{1-x^4}{1-x^2},\ \cdots$
  - We have $p_0(x)=p_d(x)$, because

$$p_d(x) = (1+x)(1+x^2)(1+x^3)\cdots = \frac{1-x^2}{1-x}\frac{1-x^4}{1-x^2}\frac{1-x^6}{1-x^3}\cdots = \frac{1}{1-x}\frac{1}{1-x^3}\cdots = p_o(x)$$

Ferrers Graph

- Use rows of dots to represent a partition of an integer
- No. element in each row is decreasing
  - (a) 4+3+3+2+1+1=14
  - (b) 6+4+3+1
- (a) and (b) are transpose
- The number of partition of n into m summands is equal to the number of partitions of n into summands where m is the largest summand

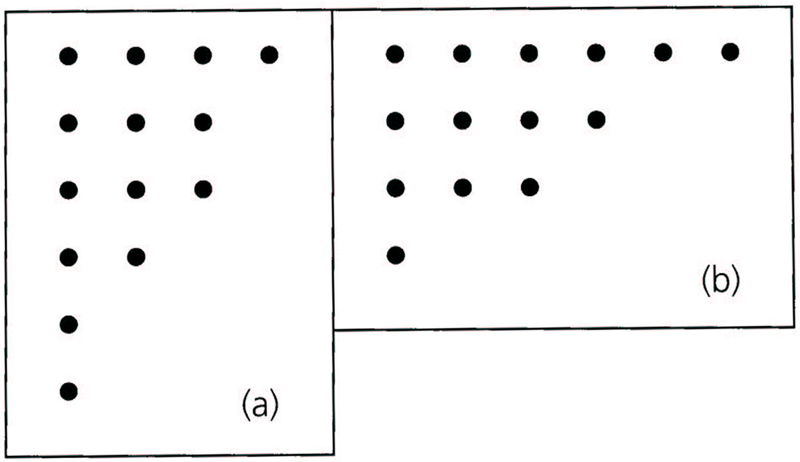

Figure 9.2

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Exponential Generating Function

- Previously seen generating functions are referred to as ordinary generating functions for given sequences
  - For selections problems ← when order is not important
  - How about arrangement problems? ← order matters
- Recall that $C(n, r) = \frac{P(n, r)}{r!}$
- Definition: For a sequence $a_0, a_1, a_2, \ldots$ of real numbers, its exponential generating function is

$$f(x) = a_0 + a_1 x + a_2 \frac{x^2}{x!} + a_3 \frac{x^3}{3!} + \cdots = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}$$

Examples

- Ex 9.25: The Maclaruin series of e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

  - e^x is the exponential generating function of 1, 1, 1,…
  - e^x is the ordinary generating function of what?

- Ex 9.26: How many ways can we arrange 4 letters out of ENGINE
  - Two E and N: $1 + x + \frac{x^2}{2!}$
  - One G and I: $1 + x$
  - The coefficient of $x^4/4!$ can be derived from

$$\frac{x^4}{2!2!} + 6\frac{x^4}{2!} + x^4 = (\frac{4!x^4}{2!2!} + 6\frac{4!x^4}{2!} + 4!)\frac{x^4}{4!}$$

More Examples

- Ex 9.27: The Maclaruin series of e^x and e^{-x} are

$$e^x = 1 + x + \frac{x^2}{2!} + x^3 3! + \cdots, \quad e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

  - Hence: $\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!}, + \cdots$ $\quad \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

- Ex 9.28: A ship with 48 flags, 12 each of them are in red, white, blue, and black. How many ways we can place 12 flags in a row so that there are even number of blue flags and odd number of black flags?
  - Exponential generating function

$$f(x) = (1 + x + \frac{x^2}{2!} + \cdots)^2 (1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots)(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots)$$

  - Simplify into

$$f(x) = (e^x)^2 (\frac{e^x + e^{-x}}{2})(\frac{e^x - e^{-x}}{2}) = \frac{1}{4}(e^{4x} - 1) = \frac{1}{4}\sum_{i=1}^{\infty} \frac{(4x)^i}{i!}$$

  - Coefficient of $\frac{x^{12}}{12!}$ is $\frac{1}{4}4^{12} = 4^{11}$

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Summation Operator

- Motivation: from $a_0, a_1, a_2, \ldots.$ to $a_0, a_0+a_1, a_0+a_1+a_2, \ldots.$
- Let $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$, what is $f(x)/(1-x)$?

$$f(x) \times \frac{1}{1-x} = (a_0 + a_1 x + a_2 x^2 + \cdots)(1 + x + x^2 + \cdots)$$
$$= a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + \cdots$$

- So we refer to $1/(1-x)$ as summation operator

Examples

- Ex 9.30 (a): 1/(1-x) is the generating function for 1, 1, 1, …. What is the sequence of $\frac{1}{1-x}\frac{1}{1-x}$?
  - 1, 2, 3, 4, …..
- Ex 9.31 (b):
  - $x+x^2$: 0, 1, 1, 0, 0, 0,…
  - $(x+x^2)/(1-x)$: 0, 1, 2, 2, 2, 2, …
  - $(x+x^2)/(1-x)^2$: 0, 1, 3, 5, 7, 9, …
  - $(x+x^2)/(1-x)^3$: 0, 1, 4, 9, 16, 25, …

Take-home Exercises

- Exercise 9.1: 1, 3, 4
- Exercise 9.2: 1, 2, 9, 18, 19
- Exercise 9.3: 1, 3, 4, 6
- Exercise 9.4: 1, 2, 5, 6, 9
- Exercise 9.5: 1, 2, 3, 5