SOLUTION

Ex 10.1: 2, 3, 7, 9

Ex 10.2: 1, 3, 4, 20, 31

Ex 10.3: 1, 2, 4, 5, 11

Ex 10.4: 1

Ex 10.1: (2)

a) $a_{n+1} = 1.5a_n, a_n = (1.5)^n a_0, n \geq 0.$

b) $4a_n = 5a_{n-1}, a_n = (1.25)^n a_0, n \geq 0.$

c) $3a_{n+1} = 4a_n, 3a_1 = 15 = 4a_0, a_0 = \frac{15}{4},$

so $a_n = \left(\frac{4}{3}\right)^n a_0 = \left(\frac{4}{3}\right)^n \left(\frac{15}{4}\right) = 5\left(\frac{4}{3}\right)^{n-1}, n \geq 0.$

d) $a_n = \left(\frac{3}{2}\right) a_{n-1}, a_n = \left(\frac{3}{2}\right)^n a_0, 81 = a_4 = \left(\frac{3}{2}\right)^4 a_0,$

so $a_0 = 16$ and $a_n = 16\left(\frac{3}{2}\right)^n, \; n \geq 0.$

Ex 10.1: (3)

- $a_{n+1} - da_n = 0, n \geq 0$, so $a_n = d^n a_0$. $\frac{153}{49} = a_3 = d^3 a_0$, $\frac{1377}{2401} = a_5 = d^5 a_0 \Rightarrow \frac{a_5}{a_3} = d^2 = \frac{9}{49}$ and $d = \pm\frac{3}{7}$.

Ex 10.1: (7)

a) $19 + 18 + 17 + \cdots + 10 = 145$

b) $9 + 8 + 7 + \cdots + 1 = 45$

Ex 10.1: (9)

a) 21345

b) 52143, 52134, 25134

c) 25134, 21534, 21354, 21345

Ex 10.2: (1.a, 1.b)

a) $a_n = 5\,a_{n-1} + 6\,a_{n-2}, n \geq 2, a_0 = 1, a_1 = 3.$
Let $a_n = cr^n, c, r \neq 0$. Then the characteristic equation is $r^2 - 5r - 6 = 0 = (r-6)(r+1)$, so $r = -1{,}6$ are the characteristic roots.
$a_n = A(-1)^n + B(6)^n.$
$1 = a_0 = A + B.$
$3 = a_1 = -A + 6B$, so $B = \frac{4}{7}$ and $A = \frac{3}{7}$.
$a_n = \left(\frac{3}{7}\right)(-1)^n + \left(\frac{4}{7}\right)6^n, n \geq 0.$

b) $a_n = 4\left(\frac{1}{2}\right)^n - 2(5)^n, n \geq 0.$

Ex 10.2: (1.c)

- $a_{n+2} + \ a_n = 0, n \geq 0, a_0 = 0, a_1 = 3.$
  With $a_n = cr^n, c, r \neq 0$. The characteristic equation $r^2 + 1 = 0$ yields the characteristic roots $\pm i$.
  Hence $a_n = A(i)^n + B(-i)^n = A\left(\cos\left(\frac{\pi}{2}\right) + isin\left(\frac{\pi}{2}\right)\right)^n +$
  $B\left(\cos\left(\frac{\pi}{2}\right) + isin\left(-\frac{\pi}{2}\right)\right)^n = Ccos\left(\frac{n\pi}{2}\right) + Dsin\left(\frac{n\pi}{2}\right)$.
  $0 = a_0 = C, 3 = a_1 = Dsin\left(\frac{\pi}{2}\right) = D,$
  so $a_n = 3\sin\left(\frac{n\pi}{2}\right), n \geq 0.$

Ex 10.2: (1.d)

- $a_n - 6a_{n-1} + 9a_{n-2} = 0, n \geq 2, a_0 = 5, a_1 = 12.$
  Let $a_n = cr^n, c, r \neq 0$. Then $r^2 + 6r + 9 = 0 = (r-3)^2$, so the characteristic roots are 3,3 and $A(3^n) + Bn(3^n)$.
  $5 = a_0 = A;\ 12a_1 = 3A + 3B = 15 + 3B, B = -1.$
  $a_n = 5(3^n) - n(3^n) = (5-n)(3^n), n \geq 0.$

Ex 10.2: (1.e)

- $a_n + 2a_{n-1} + 2a_{n-2} = 0, n \geq 2, a_0 = 1, a_1 = 3.$
  $r^2 + 2r + 2 = 0, r = -1 \pm i.$
  $(-1+i) = \sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + isin\left(\frac{3\pi}{4}\right)\right).$
  $(-1-i) = \sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) - isin\left(\frac{3\pi}{4}\right)\right).$
  $a_n = \left(\sqrt{2}\right)^n \left[Acos\left(\frac{3n\pi}{4}\right) + Bsin\left(\frac{3n\pi}{4}\right)\right].$
  $1 = a_0 = A.$
  $3 = a_1 = \sqrt{2}\left[\cos\left(\frac{3\pi}{4}\right) + Bsin\left(\frac{3\pi}{4}\right)\right] = \sqrt{2}\left[\frac{-1}{\sqrt{2}} + B\frac{1}{\sqrt{2}}\right], so\ 3 =$
  $-1 + B, B = 4.$
  $a_n = \left(\sqrt{2}\right)^n \left[\cos\left(\frac{3\pi\text{n}}{4}\right) + 4\sin\left(\frac{3\pi\text{n}}{4}\right)\right], n \geq 0.$

Ex 10.2: (3)

- $(n = 0)$: $a_2 + b\,a_1 + c\,a_0 = 0 = 4 + b(1) + c(0)$, so $b = -4$.
  $(n = 1)$: $a_3 - 4\,a_2 + c\,a_1 = 0 = 37 - 4(4) + c$, so $c = -21$.
  $a_{n+2} - 4\,a_{n+1} - 21\,a_n = 0$.
  $r^2 - 4r - 21 = 0 = (r - 7)(r + 3), r = 7, -3$.
  $a_n = A(7)^n + B(-3)^n$.
  $0 = \ a_0 = A + B \Rightarrow B = -A$.
  $1 = \ a_1 = 7A - 3B = 10A$,
  so $A = \frac{1}{10}, B = -\frac{1}{10}$ and $a_n = \frac{1}{10}[7^n - (-3)^n], n \geq 0$.

Ex 10.2: (4)

- $a_n = a_{n-1} + a_{n-2}, n \geq 2, a_0 = a_1 = 1.$
  $r^2 - r - 1 = 0, r = \frac{1 \pm \sqrt{5}}{2}.$
  $a_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n.$
  $a_0 = a_1 = 1 \Rightarrow A = \frac{1+\sqrt{5}}{2\sqrt{5}}, B = \frac{\sqrt{5}-1}{2\sqrt{5}}.$

- $a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right].$

Ex 10.2: (20.a)

- $\alpha^2 = \left[\frac{1+\sqrt{5}}{2}\right]^2 = \frac{1+2\sqrt{5}+5}{4} = \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2}$
  $= \left[\frac{1+\sqrt{5}}{2}\right] + \frac{2}{2} = \alpha + 1.$

Ex 10.2: (20.b)

- Proof: (By Mathematical Induction) For $n = 1$, we have $\alpha^n = \alpha^1 = \alpha = \alpha \cdot 1 + 0 = \alpha \mathrm{F}_1 + F_0 = \alpha \mathrm{F}_n + F_{n-1}$, so the result is true in this case. This establishes the basis step. Now we assume for an arbitrary (but fixed) positive integer k that $\alpha^k = \alpha \mathrm{F}_k + F_{k-1}$. This is our inductive step. Considering $n = k + 1$, at this time, we find that
  $\alpha^{k+1} = \alpha(\alpha^k) = \alpha[\alpha \mathrm{F}_k + F_{k-1}]$ (By the inductive step)
  $= \alpha^2 F_k + \alpha F_{k-1} = (\alpha + 1)F_k + \alpha \mathrm{F}_{k-1}$[by part (a)]
  $= \alpha(F_k + F_{k-1}) + F_k = \alpha \mathrm{F}_{k+1} + F_k$.
  Since the given result is true for $n = 1$ and the truth for $n = k + 1$ follows from that for $n = k$, it follows bt the Principle of Mathematical Induction that $\alpha^n = \alpha \mathrm{F}_n + F_{n-1}$for all $n \in Z^+$.

Ex 10.2: (31)

- Let $b_n = a_n^2, b_0 = 16, b_1 = 169$.
  This yields the linear relation $b_{n+2} - 5b_{n+1} + 4b_n = 0$ with characteristic roots $r = 4,1$, so $b_n = A(1)^n + B(4)^n$.
  $b_0 = 16, b_1 = 169 \Rightarrow A = -35, B = 51$ and
  $b_n = 51(4)^n - 35$.
  Hence $a_n =, \sqrt{51(4)^n - 35}, n \geq 0$.

Ex 10.3: (1.a, 1.b)

a) $a_{n+1} - a_n = 2n + 3, n \geq 0, a_0 = 1.$

$a_1 = a_0 + 0 + 3.$

$a_2 = a_1 + 2 + 3 = a_0 + 2 + 2(3).$

$a_3 = a_2 + 2(2) + 3 = a_0 + 2 + 2(2) + 3(3).$

$a_4 = a_3 + 2(3) + 3 = a_0 + [2 + 2(2) + 2(3)] + 4(3).$

$\ldots$

$a_n = a_0 + 2[1 + 2 + 3 + \cdots + (n-1)] + n(3) = 1 + 2\left[\frac{n(n-1)}{2}\right] + 3n = 1 + n(n-1) + 3n = n^2 + 2n + 1 = (n+1)^2, n \geq 0.$

b) $a_n = 3 + n(n-1)^2, n \geq 0.$

Ex 10.3: (1.c, 1.d)

c) $a_{n+1} - 2a_n = 5, n \geq 0, a_0 = 1.$
$a_1 = 2a_0 + 5 = 2 + 5.$
$a_2 = 2a_1 + 5 = 2^2 + 2 \cdot 5 + 5.$
$a_3 = 2a_2 + 5 = 2^3 + (2^2 + 2 + 1)5.$
$\ldots$
$a_n = 2^n + 5(1 + 2 + 2^2 + \cdots + 2^{n-1}) = 2^n + 5(2^n - 1) = 6(2^n) - 2, n \geq 0.$

d) $a_n = 2^2 + n(2^{n-1}), n \geq 0.$

Ex 10.3: (2)

- $a_n = \sum_{i=0}^{n} i^2$.
  $a_{n+1} = a_n + (n+1)^2, n \geq 0, a_0 = 0$.
  $a_{n+1} - a_n = (n+1)^2 = n^2 + 2n + 1$.
  $a_n^{(h)} = A, a_n^{(p)} = Bn + Cn^2 + Dn^3$.
  $B(n+1) + C(n+1)^2 + D(n+1)^3 = Bn + Cn^2 + Dn^3 + n^2 + 2n + 1 \Rightarrow Bn + B + Cn^2 + 2Cn + C + Dn^3 + 3Dn^2 + 3Dn + D = Bn + Cn^2 + Dn^3 + n^2 + 2n + 1$. By comparing coefficients on like powers of n, we find that $C + 3D = C + 1$, so $D = \frac{1}{3}$. Also $B + 2C + 3D = B + 2$, so $C = \frac{1}{2}$. Finally, $B + C + D = 1 \Rightarrow B = \frac{1}{6}$. So $a_n = A + \frac{1}{6}n + \frac{1}{2}n^2 + \frac{1}{3}n^3$. With $a_0 = 0$, it follows that $A = 0$ and $a_n = \frac{1}{6}n[1 + 3n + 2n^2] = \frac{1}{6}n(n+1)(2n+1), n \geq 0$.

Ex 10.3: (4)

- Let p_n be the value of the account n months after January 1 of the year the account is started.
  $p_0 = 1000.$
  $p_1 = 1000 + (.06)1000 + 200 = 1.06\, p_0 + 200.$
  $p_{n+1} = 1.06\, p_n + 200, 0 \leq n \leq 46.$
  $p_{48} = 1.06\, p_{47}.$
  $p_{n+1} - 1.06\, p_n = 200, 0 \leq n \leq 46.$
  $p_n^{(h)} = A(1.06)^n, p_n^{(p)} = C.$
  $C - 1.06C = 200 \Rightarrow C = -(20000/6).$
  $p_0 = A(1.06)^0 - (20000/6) = 1000, so\ A = (26000/6).$
  $p_n = \left(\frac{26000}{6}\right)(1.06)^n - (\frac{20000}{6}).$
  $p_{47} = (\frac{26000}{6})(1.06)^{47} - (\frac{20000}{6}) = 63685.64.$
  $p_{48} = 1.06p_{47} = 67706.78.$

Ex 10.3: (5)

a) $a_{n+2} + 3a_{n+1} + 2a_n = 3^n, n \geq 0, a_0 = 0, a_1 = 1.$
With $a_n = cr^n, c, r \neq 0$, the characteristic equation $r^2 + 3r + 2 = 0 = (r+2)(r+1)$ yields the characteristic roots $r = -1, -2$.
Hence $a_n^{(h)} = A(-1)^n + B(-2)^n$, while $a_n^{(p)} = C(3)^n$.
$C(3)^{n+2} + 3C(3)^{n+1} + 2C(3)^n = 3^n \Rightarrow 9C + 9C + 2C = 1 \Rightarrow C = \frac{1}{20}$.
$a_n = A(-1)^n + B(-2)^n + \frac{1}{20}3^n$.
$0 = a_0 = A + B + \frac{1}{20}$.
$1 = a_1 = -A - 2B + \frac{3}{20}$.
Hence $1 = a_0 + a_1 = -B + \frac{4}{20}$ and $B = -\frac{4}{5}$.
Then $A = -B - \frac{1}{20} = \frac{3}{4}$.
$a_n = \frac{3}{4}(-1)^n + -\frac{4}{5}(-2)^n + \frac{1}{20}(3)^n, n \geq 0.$

b) $a_n = \frac{2}{9}(-2)^n - \frac{5}{6}n(-2)^n + \frac{7}{9}, n \geq 0.$

Ex 10.3: (11.a)

- Let $a_n^2 = b_n, n \geq 0$.
  $b_{n+2} - 5\, b_{n+1} + 6\, b_n = 7n$.
  $b_n^{(h)} = A(3^n) + B(2^n), b_n^{(p)} = Cn + D$.
  $C(n+2) + D - 5[C(n+1) + D] + 6(Cn + D) = 7n \Rightarrow C = \frac{7}{2}, D = \frac{21}{4}$.
  $b_n = A(3^n) + B(2^n) + \frac{7n}{2} + \frac{21}{4}$.
  $b_0 = a_0^2 = 1, b_1 = a_1^2 = 1$.
  $1 = b_0 = A + B + \frac{21}{4}$.
  $1 = b_1 = 3A + 2B + \frac{7}{2} + \frac{21}{4}$.
  $3A + 2B = -\frac{31}{3}$.
  $2A + 2B = -\frac{34}{4}$.
  $A = \frac{3}{4}, B = -5$.
  $a_n = \left[\frac{3}{4} 3^n - 5 \cdot 2^n + \frac{7n}{2} + \frac{21}{4}\right]^{\frac{1}{2}}, n \geq 0$.

Ex 10.3: (11.b)

- $a_n^2 - 2a_{n-1} = 0, n \geq 1, a_0 = 2.$
  $a_n^2 = 2a_{n-1}.$
  $\log_2 a_n^2 = \log_2(2a_{n-1}) = \log_2 2 + \log_2 a_{n-1}.$
  $2\log_2 a_n = 1 + \log_2 a_{n-1}.$
  Let $b_n = \log_2 a_n.$
  The solution of the recurrence relation $2b_n = 1 + b_{n-1}$is
  $b = A\left(\frac{1}{2}\right)^n + 1.$
  $b_0 = \log_2 a_0 = \log_2 2 = 1$, so $1 = b_0 = A + 1$ and $A = 0.$
  Consequently, $b_n = 1, n \geq 0$, and $a_n = 2, n \geq 0.$

Ex 10.4: (1.a, 1.b)

a) $a_{n+1} - a_n = 3^n, n \geq 0, a_0 = 1.$

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

$\sum_{n=0}^{\infty} a_{n+1} x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=0}^{\infty} 3^n x^{n+1}$.

$[f(x) - a_0] - xf(x) = x \sum_{n=0}^{\infty} (3x)^n = \frac{x}{1-3x}$.

$f(x) - 1 - xf(x) = \frac{x}{1-3x}$.

$f(x) = \frac{1}{1-x} + \frac{x}{(1-x)(1-3x)} =$

$\frac{1}{1-x} + \left(-\frac{1}{2}\right)\left(\frac{1}{1-x}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{1-3x}\right) = \frac{1}{2}\left(\frac{1}{1-x}\right) + \frac{1}{2}\left(\frac{1}{1-3x}\right)$,

and $a_n = \frac{1}{2}[1 + 3^n], n \geq 0.$

b) $a_n = 1 + \left[\frac{n(n-1)(2n-1)}{6}\right], n \geq 0.$

Ex 10.4: (1.c)

- $a_{n+2} - 3\, a_{n+1} + 2\, a_n = 0, n \geq 0, a_0 = 1, a_1 = 6.$
  $\sum_{n=0}^{\infty} a_{n+2}x^{n+2} - 3\sum_{n=0}^{\infty} a_{n+1}x^{n+2} + 2\sum_{n=0}^{\infty} a_n x^{n+2} = 0.$
  $\sum_{n=0}^{\infty} a_{n+2}x^{n+2} - 3x\sum_{n=0}^{\infty} a_{n+1}x^{n+1} + 2x^2\sum_{n=0}^{\infty} a_n x^n = 0.$
  Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Then
  $(f(x) - 1 - 6x) - 3x(f(x) - 1) + 2x^2 f(x) = 0$, and
  $f(x)(1 + 3x + 2x^2) = 1 + 6x - 3x = 1 + 3x$, Consequently,
  $f(x) = \frac{1+3x}{(1-2x)(1-x)} = \frac{5}{1-2x} + \frac{-4}{1-x} = 5\sum_{n=0}^{\infty}(2x)^n - 4\sum_{n=0}^{\infty} x^n,$
  and $a_n = 5(2^n) - 4, n \geq 0.$

Ex 10.4: (1.d)

- $a_{n+2} - 2\,a_{n+1} + \,a_n = 2^n, n \geq 0, a_0 = 1, a_1 = 2.$
  $\sum_{n=0}^{\infty} a_{n+2}x^{n+2} - 2\sum_{n=0}^{\infty} a_{n+1}x^{n+2} + \sum_{n=0}^{\infty} a_n x^{n+2} = \sum_{n=0}^{\infty} 2^n x^{n+2}.$
  Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Then
  $[f(x) - a_0 - a_{1x}] - 2x[f(x) - a_0] + x^{2f(x)} = x^2 \sum_{n=0}^{\infty}(2x)^n$.
  $f(x) - 1 - 2x - 2xf(x) + 2x + x^2 f(x) = \frac{x^2}{1-2x}.$
  $(x^2 - 2x + 1)f(x) = 1 + \frac{x^2}{1-2x} \Rightarrow f(x) = \frac{1}{(1-x)^2} + \frac{x^2}{(1-2x)(1-x)^2} = \frac{1-2x+x^2}{(1-x)^2(1-2x)} = \frac{1}{1-2x} = 1 + 2x + (2x)^2 + \cdots$, so $a_n = 2^n, n \geq 0.$