Solution

Ex 2.1: 4, 6, 13, 17

Ex 2.2: 6, 14, 15, 19

Ex 2.3: 3, 8, 10, 12

Ex 2.4: 3, 6, 8, 19

Ex 2.5: 7, 19, 21, 24

Ex 2.1: (4)

a) $\mathrm{r} \rightarrow q$
b) $\mathrm{q} \rightarrow p$
c) $(\mathrm{s} \wedge r) \rightarrow q)$

Ex 2.1: (6)

a) True
b) False
c) True

Ex 2.1: (13)

- $(q \rightarrow [(\neg p \vee r) \wedge \neg s]) \wedge [\neg s \rightarrow (\neg r \wedge q)]: True$

  $\rightarrow (q \rightarrow [(\neg p \vee r) \wedge \neg s]): True$

  $\rightarrow [(\neg p \vee r) \wedge \neg s]: True$

  $\rightarrow s: False$

  $\rightarrow \neg p \vee r: True$

  $\rightarrow [\neg s \rightarrow (\neg r \wedge q)]: True$

  $\rightarrow (\neg r \wedge q): True$

  $\rightarrow r: False$

  $\rightarrow p: False$

Ex 2.1: (17)

Consider the following possibilities:

① Suppose that either the first or the second statement is the true one. Then statements (3) and (4) are false — so their negation are true. And we find from (3) that Tyler did not eat the piece of pie — while from (4) we conclude that Tyler did eat the pie.

② Now we'll suppose that statement (3) is the only true statement. So statements (3) and (4) no longer contract each other. But now statement (2) is false, and we have Dawn guilty (from statement (2)) and Tyler guilty (from statement (3)).

③ Finally, consider the last possibility — that is, statement (4) is the true one. Once again statement (3) and (4) do not contradict each other, and here we learn from statement (2) that Dawn is the vile culprit.

Ex 2.2: (6)

a) $\neg[p \wedge (q \vee r) \wedge (\neg p \vee \neg q \vee r)]$

$$\leftrightarrow \neg p \vee (\neg q \wedge \neg r) \vee (p \wedge q \wedge \neg r)$$
$$\leftrightarrow (\neg q \wedge \neg r) \vee [\neg p \vee (p \wedge q \wedge \neg r)]$$
$$\leftrightarrow (\neg q \wedge \neg r) \vee [T_0 \wedge (\neg p \vee (q \wedge \neg r))]$$
$$\leftrightarrow (\neg q \wedge \neg r) \vee [\neg p \vee (q \wedge \neg r)]$$
$$\leftrightarrow \neg p \vee [(\neg q \vee q) \wedge \neg r] \leftrightarrow \neg p \vee \neg r$$

b) $\neg[(p \wedge q) \rightarrow r] \leftrightarrow \neg[\neg(p \wedge q) \vee r] \leftrightarrow (p \wedge q) \wedge \neg r$

c) $p \wedge (q \vee \neg r)$

d) $\neg p \wedge \neg q \wedge \neg r$

Ex 2.2: (14.a & 14.b)

a)

p	q	$p \wedge q$	$q \rightarrow (p \wedge q)$	$p \rightarrow [q \rightarrow (p \wedge q)]$
0	0	0	1	1
0	1	0	0	1
1	0	0	1	1
1	1	1	1	1

b) Replace each occurrence of p by $p \vee q$. Then we have the tautology $(p \vee q) \rightarrow [q \rightarrow [(p \vee q) \wedge q]]$ by the first substitution rule. Since $(p \vee q) \wedge q \leftrightarrow q$, by the absorption laws, it follows that $(p \vee q) \rightarrow [q \rightarrow q] \leftrightarrow T_0$

Ex 2.2: (14.c)

p	q	$p \vee q$	$p \wedge q$	$q \rightarrow (p \wedge q)$	$(p \vee q) \rightarrow [q \rightarrow (p \wedge q)]$
0	0	0	0	1	1
0	1	1	0	0	0
1	0	1	0	1	1
1	1	1	1	1	1

The statement is not a tautology.

Ex 2.2: (15)

a) $\neg p \leftrightarrow (p \uparrow p)$

b) $p \vee q \leftrightarrow (\neg p \uparrow \neg q) \leftrightarrow (p \uparrow p) \uparrow (q \uparrow q)$

c) $p \wedge q \leftrightarrow \neg\neg(p \wedge q) \leftrightarrow \neg(p \uparrow q) \leftrightarrow (p \uparrow q) \uparrow (p \uparrow q)$

d) $(p \rightarrow q) \leftrightarrow \neg p \vee q \leftrightarrow \neg(p \wedge \neg q) \leftrightarrow (p \uparrow \neg q) \leftrightarrow p \uparrow (q \uparrow q)$

e) $(p \leftrightarrow q) \leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p) \leftrightarrow t \wedge u \leftrightarrow (t \uparrow u) \uparrow (t \uparrow u)$, where t stands for $p \uparrow (q \uparrow q)$ and u for $q \uparrow (p \uparrow p)$.

Ex 2.2: (19.a & 19.b)

(a) $\mathbf{p} \vee [\mathbf{p} \wedge (\mathbf{p} \vee \mathbf{q})]$	**Reasons**
$\leftrightarrow p \vee p$	Absorption Law
$\leftrightarrow p$	Idempotent

(b) $\mathbf{p} \vee \mathbf{q} \vee (\neg \mathbf{p} \wedge \neg \mathbf{q} \wedge \mathbf{r})$	**Reasons**
$\leftrightarrow (p \vee q) \vee (\neg(p \vee q) \wedge r)$	DeMorgan's Laws
$\leftrightarrow [(p \vee q) \vee \neg(p \vee q)] \wedge (p \vee q \vee r)$	Distributive Law of ∨ over ∧
$\leftrightarrow T0 \wedge (p \vee q \vee r)$	Inverse Law
$\leftrightarrow p \vee q \vee r$	Identity Law

Ex 2.2: (19.c)

(c) $(\neg \mathbf{p} \vee \neg \mathbf{q}) \rightarrow (\mathbf{p} \wedge \mathbf{q} \wedge \mathbf{r})$	**Reasons**
$\leftrightarrow \neg(\neg p \vee \neg q) \vee (p \wedge q \wedge r)$	$s \rightarrow t \leftrightarrow \neg s \vee t$
$\leftrightarrow (\neg\neg p \wedge \neg\neg q) \vee (p \wedge q \wedge r)$	DeMorgan's Laws
$\leftrightarrow (p \wedge q) \vee (p \wedge q \wedge r)$	Law of Double Negation
$\leftrightarrow p \wedge q$	Absorption Law

Ex 2.3: (3)

a) If p has the truth value 0, then so does $p \wedge q$.
b) When $p \vee q$ has the truth value 0, then the truth value of p (and that of q) is 0.
c) If q has the truth value 0, then the truth value of $[(p \vee q) \wedge \neg p]$ is 0, regardless of the truth value of p.
d) The statement $q \vee s$ has the truth value 0 only when each of q, s has the truth value 0. Then $(p \rightarrow q)$ has truth value 1 when p has the truth value 0; $(r \rightarrow s)$ has truth value 1 when r has truth value 0. But then $(p \vee r)$ must have truth value 0, not 1.
e) For $(\neg p \vee \neg r)$ the truth value is 0 when both p, r have truth value 1. This then forces q, s to have truth value 1, in order for $(p \rightarrow q)$, $(r \rightarrow s)$ to have truth value 1. However, this results in truth value 0 for $(\neg q \rightarrow \neg s)$.

Ex 2.3: (8)

1) Premise
2) Step (1) and the Rule of Conjunctive Simplification
3) Premise
4) Steps (2), (3) and the Rule of Detachment
5) Step (1) and the Rule of Conjunctive Simplification
6) Steps (4), (5) and the Rule of Conjunction
7) Premise
8) Step (7) and $[r \rightarrow (s \vee t)] \leftrightarrow [\neg(s \vee t) \rightarrow r]$
9) Step (8) and DeMorgan's Laws
10) Steps (6), (9) and the Rule of Detachment
11) Premise
12) Step (11) and $[(\neg p \vee q) \rightarrow r] \leftrightarrow [\neg r \rightarrow \neg(\neg p$
13) Step (12) and DeMorgan's Laws and the Law of Double Negation
14) Steps (10), (13) and the Rule of Detachment
15) Step (14) and the Rule of Conjunctive Simplification

Ex 2.3: (10.a & 10.b)

(a)	**Reasons**
1) $p \wedge \neg q$	1) Premise
2) p	2) Step (1) and the Rule of Conjunctive Simplification
3) r	3) Premise
4) $p \wedge r$	4) Steps (2), (3) and the Rule of Conjunction
5) $\therefore (p \wedge r) \vee q$	5) Step (4) and the Rule of Disjunctive Amplification

(b)	**Reasons**
1) $p, p \rightarrow q$	1) Premise
2) q	2) Step (1) and the Rule of Detachment
3) $\neg q \vee r$	3) Premise
4) $q \rightarrow r$	4) Step (3) and $(\neg q \vee r) \leftrightarrow (q \rightarrow r)$
5) $\therefore r$	5) Steps (2), (4) and the Rule of Detachment

Ex 2.3: (10.c & 10.d)

(c)	**Reasons**
1) $p \rightarrow q, \neg q$	1) Premises
2) $\neg p$	2) Step (1) and Modus Tollens
3) $\neg r$	3) Premise
4) $\neg p \wedge \neg r$	4) Steps (2), (3) and the Rule of Conjunction
5) $\therefore \neg(p \vee r)$	5) Step (4) and DeMorgan's Laws

(d)	**Reasons**
1) $r, r \rightarrow \neg q$	1) Premises
2) $\neg q$	2) Step (1) and the Rule of Detachment
3) $p \rightarrow q$	3) Premise
4) $\neg p$	4) Steps (2), (3) Modus Tollens

Ex 2.3: (10.e)

(e)

1) p
2) $\neg q \rightarrow \neg p$
3) $p \rightarrow q$
4) q
5) $p \wedge q$
6) $p \rightarrow (q \rightarrow r)$
7) $(p \wedge q) \rightarrow r$
8) $\therefore r$

Reasons

1) Premise
2) Premise
3) Step (2) and $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$
4) Steps (1), (3) and the Rule of Detachment
5) Steps (1), (4) and the Rule of Conjunction
6) Premise
7) Step (6), and $[p \rightarrow (q \rightarrow r)] \leftrightarrow [(p \wedge q) \rightarrow r]$
8) Steps (5), (7) and the Rule of Detachment

Ex 2.3: (10.f)

(f)	**Reasons**
1) $p \wedge q$	1) Premise
2) p	2) Step (1) and the Rule of Conjunctive Simplification
3) $p \rightarrow (r \wedge q)$	3) Premise
4) $r \wedge q$	4) Step (2), (3) and the Rule of Detachment
5) r	5) Steps (4) and the Rule of Conjunctive Simplification
6) $r \rightarrow (s \vee t)$	6) Premise
7) $s \vee t$	7) Steps (5), (6) and the Rule of Detachment
8) $\neg s$	8) Premise
9) $\therefore r$	9) Steps (7), (8) and the Rule of Disjunctive Syllogism

Ex 2.3: (10.g)

(g)

1) $\neg s, p \vee s$
2) p
3) $p \to (q \to r)$
4) $q \to r$
5) $t \to q$
6) $t \to r$
7) $\therefore \neg r \to \neg t$

Reasons

1) Premise
2) Step (1) and the Rule of Disjunctive Syllogism
3) Premise
4) Step (2), (3) and the Rule of Detachment
5) Premise
6) Steps (4), (5) and the Law of the Syllogism
7) Step (6) and $(t \to r) \leftrightarrow (\neg r \to \neg t)$

Ex 2.3: (10.h)

(h)	**Reasons**
1) $\neg p \vee r$	1) Premise
2) $p \rightarrow r$	2) Step (1) and $(p \rightarrow r) \leftrightarrow (\neg p \vee r)$
3) $\neg r$	3) Premise
4) $\neg p$	4) Step (2), (3) and Modus Tollens
5) $p \vee q$	5) Premise
6) $\neg p \rightarrow q$	6) Steps (5) and $(p \vee q) \leftrightarrow (\neg\neg p \vee q) \leftrightarrow (\neg p \rightarrow q)$
7) $\therefore q$	7) Steps (4), (6) and Modus Ponens

Ex 2.3: (12.a)

p: Rochelle gets the supervisor's position
q: Rochelle works hard.
r: Rochelle gets a raise
s: Rochelle buys a new car

$(p \wedge q) \rightarrow r$
$r \rightarrow s$
$\neg s$
$-------\ -$
$\therefore \neg p \vee \neg q$

1) $\neg s$	1) Premise
2) $r \rightarrow s$	2) Premise
3) $\neg r$	3) Steps (1), (2) and Modus Tollens
4) $(p \wedge q) \rightarrow r$	4) Premise
5) $\neg(p \wedge q)$	5) Steps (3), (4) and Modus Tollens
6) $\therefore \neg p \vee \neg q$	6) Step (5) and $\neg(p \wedge q) \leftrightarrow \neg p \vee \neg q$

Ex 2.3: (12.b)

p: Dominic goes to the racetrack.
q: Helen gets mad.
r: Ralph plays cards all night.
s: Carmela gets mad.
t: Veronica is notified.

$p \rightarrow q$
$r \rightarrow s$
$(q \vee s) \rightarrow t$
$\neg t$
$\overline{\quad\quad\quad\quad\quad}$
$\therefore \neg p \wedge \neg r$

1) $\neg t$	1) Premise
2) $(q \vee s) \rightarrow t$	2) Premise
3) $\neg(q \vee s)$	3) Steps (1), (2) and Modus Tollens
4) $\neg q \wedge \neg s$	4) Step (3) and $\neg(q \vee s) \leftrightarrow \neg q \wedge \neg s$
5) $\neg q$	5) Step (4) and the Rule of Conjunctive Simplification
6) $p \rightarrow q$	6) Premise
7) $\neg p$	7) Steps (5), (6) and Modus Tollens
8) $\neg s$	8) Step (4) and the Rule of Conjunctive Simplification
9) $r \rightarrow s$	9) Premise
10) $\neg r$	10) Steps (8), (9) and Modus Tollens
11) $\therefore \neg p \wedge \neg r$	11) Steps (7), (10) and the Rule of Conjunction

Ex 2.3: (12.c)

p: There is a chance of rain.
q: Lois' red head scarf is missing.
r: Lois does not mow her lawn.
s: The temperature is over 80º F.

$(p \vee q) \rightarrow r$
$s \rightarrow \neg p$
$s \wedge \neg q$
$------- -$
$\therefore \neg r$

The following truth value assignments
provide a counterexample to the validity of this argument:
p: 0; q: 0; r: 1; s: 1

Ex 2.4: (3)

- True: a, c, e
- False: b, d, f

Ex 2.4: (6)

- True: a, b, d
- False: c, e, f

Ex 2.4: (8)

a) True
b) False: For x = 1, q(x) is true while p(x) is false.
c) True
d) True
e) True
f) True
g) True
h) False: For x = -1, $(p(x) \vee q(x))$ is true but r(x) is false.

Ex 2.4: (19.a)

Statement	For all positive integers m, n, if $m > n$ then $m^2 > n^2$.	True
Converse	For all positive integers m, n, if $m^2 > n^2$ then $m > n$.	True
Inverse	For all positive integers m, n, if $m \leq n$ then $m^2 \leq n^2$.	True
Contrapositive	For all positive integers m, n, if $m^2 \leq n^2$ then $m \leq n$.	True

Ex 2.4: (19.b)

Statement	For all integers a, b, if $a > b$ then $a^2 > b^2$.	False: Let $a = 1$ and $b = -2$
Converse	For all integers a, b, if $a^2 > b^2$ then $a > b$.	False: Let $a = -5$ and $b = 3$
Inverse	For all integers a, b, if $a \leq b$ then $a^2 \leq b^2$.	False: Let $a = -5$ and $b = 3$
Contrapositive	For all integers a, b, if $a^2 \leq b^2$ then $a \leq b$.	False: Let $a = 1$ and $b = -2$

Ex 2.4: (19.c)

Statement	For all integers m, n, and p, if m divides n and n divides p, then m divides p.	True
Converse	For all integers m and p, if m divides p, then for each integer n it follows that m divides n and n divides p.	False: Let m = 1, n = 2, and p = 3
Inverse	For all integers m, n, and p, if m does not divide n or n does not divide p, then m does not divide p.	False: Let m = 1, n = 2, and p = 3
Contrapositive	For all integers m, n, and p, if m does not divide p then for each integer n it follows that m does not divide n or n does not divide p.	True

Ex 2.4: (19.d)

Statement	$\forall x[(x > 3) \rightarrow (x^2 > 9)]$	True
Converse	$\forall x[\ (x^2 > 9)] \rightarrow (x > 3)]$	False: Let x = -5
Inverse	$\forall x[(x \leq 3) \rightarrow (x^2 \leq 9)]$	False: Let x = -5
Contrapositive	$\forall x[\ (x^2 \leq 9)] \rightarrow (x \leq 3)]$	True

Ex 2.4: (19.e)

Statement	$\forall x[(x^2 + 4x - 21 > 0) \rightarrow [(x > 3) \vee (x < -7)]]$	True
Converse	$\forall x[[(x > 3) \vee (x < -7)] \rightarrow (x^2 + 4x - 21 > 0)]$	True
Inverse	$\forall x[(x^2 + 4x - 21 \leq 0) \rightarrow (-7 \leq x \leq 3)]$	True
Contrapositive	$\forall x[(-7 \leq x \leq 3) \rightarrow (x^2 + 4x - 21 \leq 0)]$	True

Ex 2.5: (7.a)

- When the statement $\exists x[p(x) \vee q(x)]$ is true, there is at least one element c in the prescribed universe where $\mathrm{p}(c) \vee q(c)$ is true. Hence at least one statements $\mathrm{p}(c),\ q(c)$ has the truth value 1, so at least one of the statement $\exists x\, p(x)$ and $\exists x\, q(x)$ is true. Therefore, it follows that $\exists x\, p(x) \vee \exists x\, q(x)$ is true, and $\exists x\, [p(x) \vee q(x)] \rightarrow \exists x\, p(x) \vee \exists x\, q(x)$. Conversely, if $\exists x\, p(x) \vee \exists x\, q(x)$ is true, then at least one of $p(a)$, $\mathrm{q}(a)$ has truth value 1, for some a, b in the prescribed universe. Assume without loss of generality that it is $p(a)$. Then $p(a) \vee q(a)$ has truth value 1 so $\exists x\, [p(x) \vee q(x)]$ is a true statement, and $\exists x\, p(x) \vee \exists x\, q(x) \rightarrow \exists x\, [p(x) \vee q(x)]$

Ex 2.5: (7.b)

* First consider when the statement $\forall x[p(x) \wedge q(x)]$ is true. This occurs when $p(a) \wedge q(a)$ is true for each a in the prescribed universe. Then $p(a)$ is true (as is $q(a)$) for all a in the universe, so the statements $\forall x\, p(x)$, $\forall x\, q(x)$ are true. Therefore, the statement $\forall x\, p(x) \wedge \forall x\, q(x)$ is true and $\forall x[p(x) \wedge q(x)] \rightarrow \forall x\, p(x) \wedge \forall x\, q(x)$. Conversely, suppose that $\forall x\, p(x) \wedge \forall x\, q(x)$ is a true statement. Then $\forall x\, p(x)$, $\forall x\, q(x)$ are both true. So now let c be any element in the prescribed universe. Then p(c), $q(c)$, and p(c) ∧ q(c) are all true. And, since c was chosen arbitrarily, it follows that the statement $\forall x[p(x) \wedge q(x)]$ is true, and $\forall x\, p(x) \wedge \forall x\, q(x) \rightarrow \forall x[p(x) \wedge q(x)]$.

Ex 2.5: (19)

- This result is not true, in general. For example, $m = 4 = 2^2$ and $\mathrm{n} = 1 = 1^2$ are two positive integers that are perfect squares, but $\mathrm{m} + \mathrm{n} = 2^2 + 1^2 = 5$ is not a perfect square.

Ex 2.5: (21)

* Proof:
  We shall prove the given result by establishing the truth of its (logically equivalent) contrapositive.
  Let us consider the negation of the conclusion --- that is, $x < 50$ and $y < 50$. Then with $x < 50$ and $y < 50$ it follows that $x + y < 50 + 50 = 100$, and we have the negation of the hypothesis. The given result now follows by this indirect method of proof (by the contrapositive).

Ex 2.5: (24)

* Proof:
  If n is even, then n = 2k for some (particular) integer k. Then 31n + 12 = 31(2k) + 12 = 2(31k + 6), so it follows from Definition 2.8 that 31n + 12 is even.
  Conversely, suppose that n is not even. Then n is odd, so n = 2t + 1 for some (particular) integer t. Therefore, 31n + 12 = 31(2t + 1) + 12 = 2(31t + 21) + 1, so from Definition 2.8 we have 31n + 12 odd --- hence, not even. Consequently, the converse follows by contraposition.