SOLUTION

Ex 7.1: 1, 5, 6, 9, 17

Ex 7.2: 4, 14, 17, 18, 26

Ex 7.3: 1, 7, 18, 23, 25

Ex 7.4: 2, 6, 7, 12, 14

Ex 7.5: 1, 3

Ex 7.1: (1)

a) $\{(1,1), (2,2), (3,3), (4,4), (1,2), (2,1), (2,3), (3,2)\}$

b) $\{(1,1), (2,2), (3,3), (4,4), (1,2)\}$

c) $\{(1,1), (2,2), (1,2), (2,1)\}$

Ex 7.1: (5)

a) reflexive, antisymmetric, transitive
b) transitive
c) reflexive, symmetric, transitive
d) symmetric
e) (odd): symmetric
f) (even): reflexive, symmetric, transitive
g) reflexive, symmetric
h) reflexive, transitive

Ex 7.1: (6)

- The relation in part (a) is a partial order.
  The relation in parts (c) and (f) are equivalence relations.

Ex 7.1: (9)

a) False: Let $A = \{1,2\}$ and $\mathcal{R} = \{(1,2),(2,1)\}$.

b) (i) Reflexive: True.
 (ii) Symmetric: False.
 Let $A = \{1,2\}$, $\mathcal{R}_1 = \{(1,1)\}$, $\mathcal{R}_2 = \{(1,1),(1,2)\}$
 (iii) Antisymmetric and transitive: False.
 Let $A = \{1,2\}$, $\mathcal{R}_1 = \{(1,2)\}$, $\mathcal{R}_2 = \{(1,2),(2,1)\}$

c) (i) Reflexive: False.
 Let $A = \{1,2\}$, $\mathcal{R}_1 = \{(1,1)\}$, $\mathcal{R}_2 = \{(1,1),(2,2)\}$
 (ii) Symmetric: False.
 Let $A = \{1,2\}$, $\mathcal{R}_1 = \{(1,2)\}$, $\mathcal{R}_2 = \{(1,2),(2,1)\}$
 (iii) Antisymmetric: True.
 (iv) Transitive: False.
 Let $A = \{1,2\}$, $\mathcal{R}_1 = \{(1,2),(2,1)\}$,
 $\mathcal{R}_2 = \{(1,1),(1,2),(2,1),(2,2)\}$

d) True.

Ex 7.1: (17)

a) $\binom{7}{4}\binom{21}{0} + \binom{7}{2}\binom{21}{1} + \binom{7}{0}\binom{21}{2}$

b) $\binom{7}{5}\binom{21}{0} + \binom{7}{3}\binom{21}{1} + \binom{7}{1}\binom{21}{2}$

c) $\binom{7}{7}\binom{21}{0} + \binom{7}{5}\binom{21}{1} + \binom{7}{3}\binom{21}{2} + \binom{7}{1}\binom{21}{3}$

d) $\binom{7}{6}\binom{21}{1} + \binom{7}{4}\binom{21}{2} + \binom{7}{2}\binom{21}{3} + \binom{7}{0}\binom{21}{4}$

Ex 7.2: (4)

a) $\mathcal{R}_1 \circ (\mathcal{R}_2 \cup \mathcal{R}_3) = \{(1,4),(1,5),(3,4),(3,5),(2,6),(1,6)\}$
$(\mathcal{R}_1 \circ \mathcal{R}_2) \cup (\mathcal{R}_1 \circ \mathcal{R}_3) = \{(1,4),(1,5),(1,6),(2,6),(3,4),(3,5)\}$

b) $\mathcal{R}_1 \circ (\mathcal{R}_2 \cap \mathcal{R}_3) = \{(1,5),(3,5)\}$
$(\mathcal{R}_1 \circ \mathcal{R}_2) \cap (\mathcal{R}_1 \circ \mathcal{R}_3) = \{(1,4),(1,5),(3,5)\}$

Ex 7.2: (14.1)

```
10!    THIS PROGRAM MAY BE USED TO DETERMINE IF A RELATION
20!    ON A SET OF SIZE N, WHERE N <= 20, IS AN
30!    EQUIVALENCE RELATION. WE ASSUME WITHOUT LOSS OF
40!    GENERALITY THAT THE ELEMENTS ARE 1,2,3,...,N.
50!
60     INPUT “N =”; N
70     PRINT “   INPUT THE RELATION MATRIX FOR THE RELATION”
80     PRINT “BEING EXAMINED BY TYPING A(I,J) = 1 FOR EACH”
90     PRINT “1 <=  I <= N, 1 <=  J <= N, WHERE (I,J) IS IN”
100    PRINT “THE RELATION. WHEN ALL THE ORDERED PAIRS HAVE”
110    PRINT “BEEN ENTERED TYPE ’CONT’ ”
120    STOP
130    DIM A(20,20), C(20,20), D(20,20)
140    FOR K = 1 TO N
150           T = T + A(K,K)
160    NEXT K
170    IF T = N THEN &
                 PRINT “R IS REFLEXIVE”; X = 1: GO TO 190
180    PRINT “R IS NOT REFLEXIVE”
```

Ex 7.2: (14.2)

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190    FOR I = 1 TO N
200        FOR J = I + 1 TO N
210            IF A(I,J) <> A(J,I) THEN GO TO 260
220        NEXT J
230    NEXT I
240    PRINT "R IS SYMMETRIC": Y = 1
250    GO TO 270
260    PRINT "R IS NOT SYMMETRIC"
270    MAT C = A
280    MAT D = A*C
290    FOR I = 1 TO N
300        FOR J = 1 TO N
310            IF D(I,J) > 0 AND A(I,J) = 0 THEN GO TO 360
320        NEXT J
330    NEXT I
340    PRINT "R IS TRANSITIVE"; Z = 1
350    GO TO 370
360    PRINT "R IS NOT TRANSITIVE"
370    IF X + Y + Z = 3 THEN &
                   PRINT "R IS AN EQUIVALENCE RELATION" &
       ELSE PRINT "R IS NOT AN EQUIVALENCE RELATION"
380    END
```

Ex 7.2: (17.i)

- $\mathcal{R} = \{(a,b),(b,a),(a,e),(e,a),(b,c),(c,b),$
  $(b,d),(d,b),(b,e),(e,b),(d,e),(e,d),(d,f),(f,d)\}$

- $M(\mathcal{R}) = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$

Ex 7.2: (17.ii)

- $\mathcal{R} = \{(a, b), (b, e), (d, b), (d, c), (e, f)\}$
- $M(\mathcal{R}) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Ex 7.2: (17.iii)

- $\mathcal{R} = \{(a,a),(a,b),(b,a),(c,d),(d,c),(d,e),$
  $(e,d),(d,f),(f,d),(e,f),(f,e)\}$

- $M(\mathcal{R}) = \begin{bmatrix} 1&1&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&1&0&1&1 \\ 0&0&0&1&0&1 \\ 0&0&0&1&1&0 \end{bmatrix}$

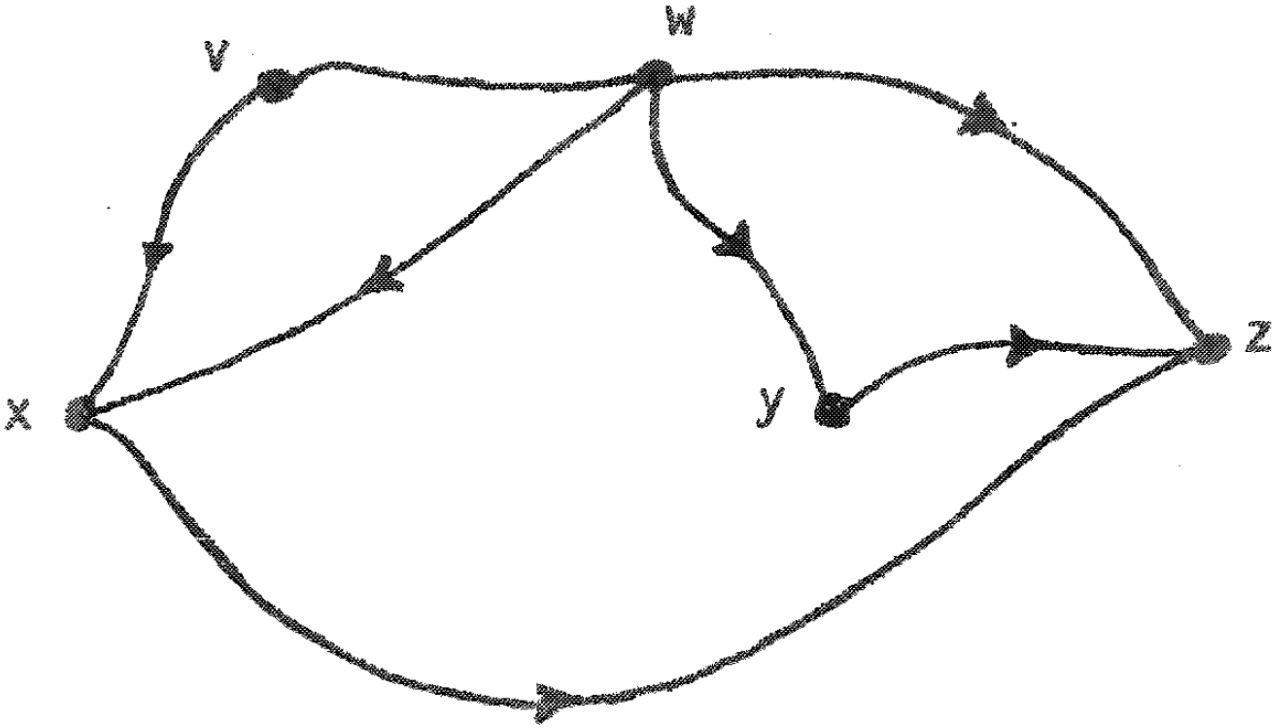
Ex 7.2: (17.iv)

- $\mathcal{R} = \{(b,a),(b,c),(c,b),(b,e),(c,d),(e,d)\}$

- $M(\mathcal{R}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

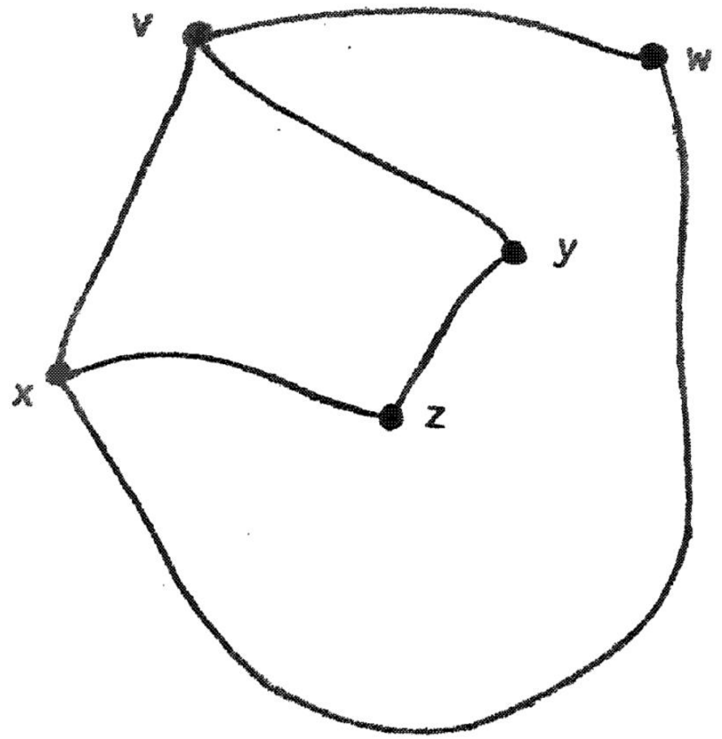
Ex 7.2: (18.a)

- $\mathcal{R} = \{(v,w),(v,x),(w,v),(w,x),(w,y),(w,z),(x,z),(y,z)\}$



Ex 7.2: (18.b)

- $\mathcal{R} = \{(v,w),(v,x),(v,y),(w,v),(w,x),(x,v),$
  $(x,w),(x,z),(y,v),(y,z),(z,x),(z,y)\}$



Ex 7.2: (26.a, 26.b)

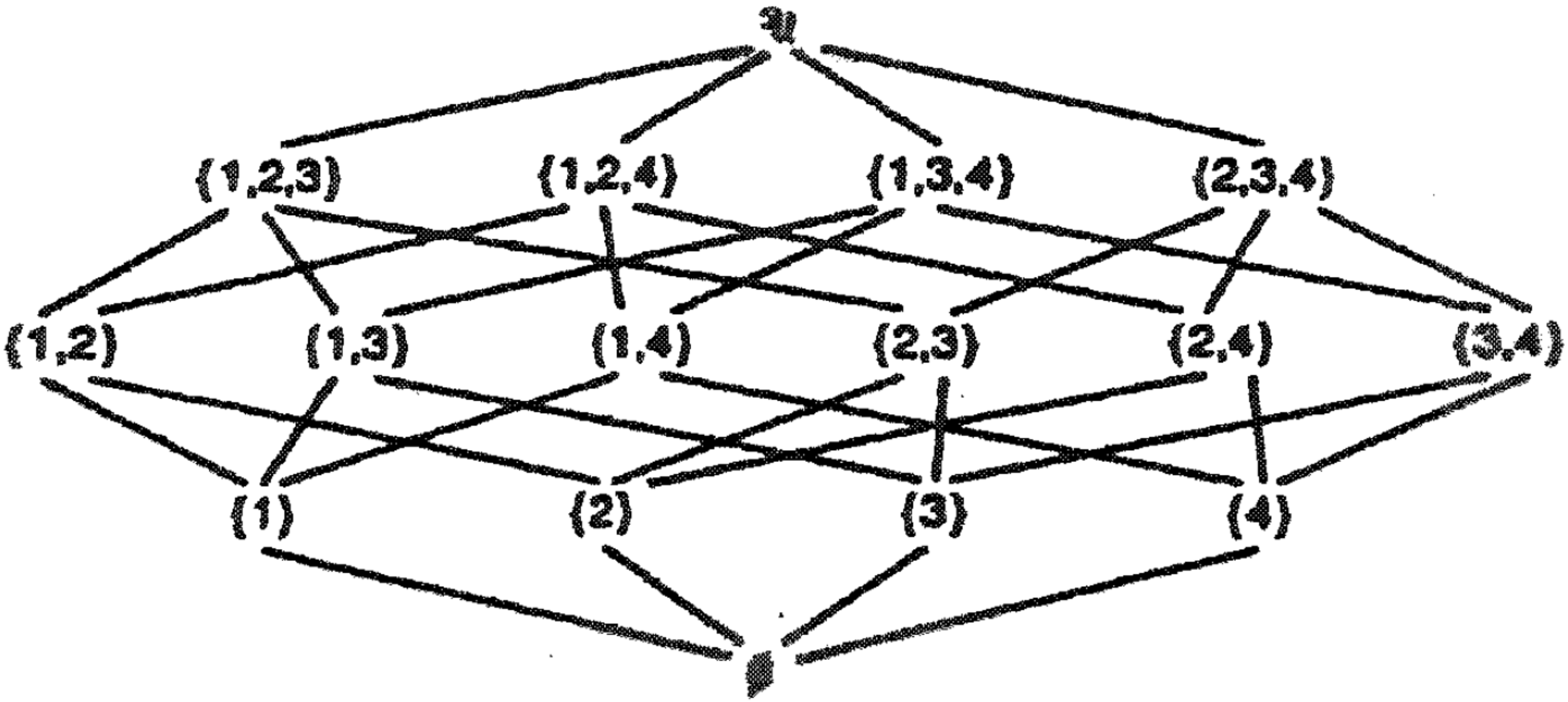
a) Let $k \in \mathrm{Z}^{+}$. Then $R^{12k} = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(7,7)\}$ and $R^{12k+1} = R$. The smallest value of $n > 1$ such that $R^{n} = R$ is $n = 13$. When $n = 3, (5,5)(6,6)(7,7) \in \mathrm{R}^{3}$, and this is the smallest power of R that contains at least one loop. For all multiples of 12 the graph consists of only loops.

b) For all $k \in \mathrm{Z}^{+}$, $R^{30k} = \{(x,x) | x \in \mathrm{Z}^{+}, 1 \leq\ x \leq 10\}$ and $R^{30k+1} = R$. Hence R^{31} is the smallest power of R (for $n > 1$) where $R^{n} = R$. When $n = 2$, we find $(1,1),(2,2)$ in R, which is the smallest n with loops in the graph.

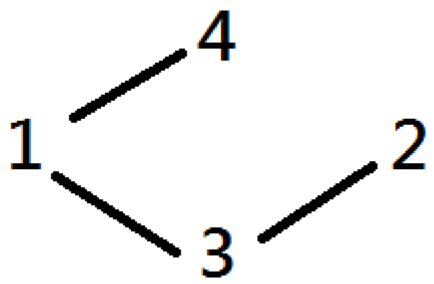
Ex 7.2: (26.c)

- Let R be a relation on set A where $|A| = m$. Let G be the directed graph associated with R – each component of G is a directed cycle C_i on m_i vertices, with $1 \leq \ i \leq \ k$. (Thus $m_1 + m_2 + \cdots + m_k = m$.) The smallest power of R where loops appear is R^t, for $t = \min\{m_i | 1 \leq \ i \leq \ k\}$.
- Let $s = lcm(m_1, m_2, \dots, m_k)$. Then R^{rs} =the identity (equality) relation on A and $R^{rs+1} = R$, for all $r \in \mathrm{Z}^+$. The smallest power of R that reproduces R is $s + 1$.

Ex 7.3: (1)



Ex 7.3: (7)

a) 

b) $3 < 2 < 1 < 4 \ or \ 3 < 1 < 2 < 4$

c) 2

Ex 7.3: (18)

a) (i) Only one such upper bound – $\{1,2,3\}$.
(ii) Here the upper bound has the form $\{1,2,3,x\}$ where $x \in \mathcal{U}$ and $4 \leq x \leq 7$. Hence there are four such upper bounds.
(iii) There are $\binom{4}{2}$ upper bounds of B that contain five elements from $\mathcal{U}$.

b) $\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 2^4 = 16$

c) $lub\ B = \{1,2,3\}$

d) One–namely $\emptyset$

e) $glb\ B = \emptyset$

Ex 7.3: (23)

a) Flase. Let $U = \{1,2\}, A = P(\mathcal{U})$, and $\mathcal{R}$ be the inclusion relation. Then $(A, \mathcal{R})$ is a lattice where for all $S, T \in \mathrm{A}$, $lub\{S, T\} = S \cup \mathrm{T}$ and $glb\{S, T\} = S \cap \mathrm{T}$. However, $\{1\}$ and $\{2\}$ are not related, so $(A, \mathcal{R})$ is not a total order.

b) If $(A, \mathcal{R})$ is a total order, then for all $x, y \in \mathrm{A}, x\mathcal{R}y$ or $y\mathcal{R}x$. For $x\mathcal{R}y, lub\{x, y\} = y$ and $glb\{x, y\} = x$. Consequently, $(A, \mathcal{R})$ is a lattice.

Ex 7.3: (25)

a) a
b) a
c) c
d) e
e) z
f) e
g) v

$(A, \mathcal{R})$ is a lattice with z the greatest (and only maximal) element and a the least (and only minimal) element.

Ex 7.4: (2)

a) There are three choices for placing 8 – in either A_1, A_2 or A_3. Hence there are three partitions of A for the conditions given.
b) There are two possibilities with $7 \in A_1$, and two others with $8 \in A_1$. Hence there are four partitions of A under these conditions.
c) If we place $7{,}8$ in the same cell for a partition we obtain three of the possibilities. If not, there are three choices of cells for 7 and two choices of cells for 8 – and six more partitions that satisfy the stated restrictions. In total – by the rules of sum and produce – there are $3+(3)(2)=3+6=9$ such partitions.

Ex 7.4: (6)

a) For all $(x, y) \in A$, since $x = x$, it follows that $(x, y)R(x, y)$, so R is *reflexive*.
If $(x_1, y_1), (x_2, y_2) \in$ A and $(x_1, y_1)R(x_2, y_2)$, then $x_1 = x_2$, so $x_2 = x_1$ and $(x_2, y_2)R(x_1, y_1)$. Hence R is *symmetric*.
Finally, let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in$ A with $(x_1, y_1)R(x_2, y_2)$ and $(x_2, y_2)R(x_3, y_3)$. $(x_1, y_1)R(x_2, y_2) \Rightarrow x_1 = x_2$; $(x_2, y_2)R(x_3, y_3) \Rightarrow x_2 = x_3$. With $x_1 = x_2, x_2 = x_3$, it follows that $x_1 = x_3$, so $(x_1, y_1)R(x_3, y_3)$ and R is *transitive*.

b) Each equivalence class consists of the points on a vertical line. The collection of these vertical lines then provides a partition of the real plane.

Ex 7.4: (7)

a) For all $(x, y) \in \mathrm{A}, x + y = x + y \Rightarrow (x, y)R(x, y)$.
$(x_1, y_1)R(x_2, y_2) \Rightarrow\ x_1 + y_1 = x_2 + y_2 \Rightarrow x_2 + y_2 = x_1 + y_1$
$\Rightarrow (x_2, y_2)R(x_1, y_1)$. $(x_1, y_1)R(x_2, y_2), (x_2, y_2)R(x_3, y_3)$
$\Rightarrow\ x_1 + y_1 = x_2 + y_2, x_2 + y_2 = x_3 + y_3$, so $x_1 + y_1 = x_3 + y_3$
and $(x_1, y_1)R(x_3, y_3)$.
Since R is reflexive, symmetric and transitive, it is an equivalence relation.

b) $[(1,3)] = \{(1,3), (2,2), (3,1)\}$;
$[(2,4)] = \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$; $[(1,1)] = \{(1,1)\}$.

c) $A = \{(1,1)\} \cup \{(1,2), (2,1)\} \cup \{(1,3), (2,2), (3,1)\} \cup$
$\{(1,4), (2,3), (3,2), (4,1)\} \cup \{(1,5), (2,4), (3,3), (4,2), (5,1)\} \cup$
$\{(2,5), (3,4), (4,3), (5,2)\} \cup \{(3,5), (4,4), (5,3)\} \cup \{(4,5), (5,4)\} \cup$
$\{(5,5)\}$.

Ex 7.4: (12)

a) $2^{10} = 1024$

b) $\sum_{i=1}^{5} S(5, i) = 1 + 15 + 25 + 10 + 1 = 52$

c) $1024 - 52 = 972$

d) $S(5,2) = 15$

e) $\sum_{i=1}^{4} S(4, i) = 1 + 7 + 6 + 1 = 15$

f) $\sum_{i=1}^{3} S(3, i) = 1 + 3 + 1 = 5$

g) $\sum_{i=1}^{3} S(3, i) = 1 + 3 + 1 = 5$

h) $(\sum_{i=1}^{3} S(3, i)) - (\sum_{i=1}^{2} S(2, i)) = 3$

Ex 7.4: (14)

a) Not possible. With R reflexive, $|R| \leq 7$.

b) $R = \{(x, x) | x \in \mathrm{Z}, 1 \leq x \leq 7\}$.

c) Not possible. With R symmetric, $|R| - 7$ must be even.

d) $R = \{(x, x) | x \in \mathrm{Z}, 1 \leq x \leq 7\} \cup \{(1,2), (2,1)\}$.

e) $R = \{(x, x) | x \in \mathrm{Z}, 1 \leq x \leq 7\}$
$\cup \{(1,2), (2,1)\} \cup \{(3,4), (4,3)\}$.

f) Not possible with $r - 7$ odd.

g) Not possible. See the remark at the end of Section 7.4.

h) Not possible with $r - 7$ odd.

i) Not possible. See the remark at the end of Section 7.4.

Ex 7.5: (1)

a) $P_1: \{s_1, s_4\}, \{s_2, s_3, s_5\}$
$(v(s_1, 0) = s_4) E_1 (v(s_4, 0) = s_1)$ but
$(v(s_1, 1) = s_1) \not{E}_1 (v(s_4, 1) = s_3)$, so $s_1 \not{E}_2 s_4$.
$(v(s_2, 1) = s_3) \not{E}_1 (v(s_3, 1) = s_4)$, so $s_2 \not{E}_2 s_3$.
$(v(s_2, 0) = s_3) E_1 (v(s_5, 1) = s_3)$ and
$(v(s_2, 1) = s_3) E_1 (v(s_5, 1) = s_3)$, so $s_2 \not{E}_2 s_5$.
Since $s_2 \not{E}_2 s_3$ and $s_2 E_2 s_5$, it follows that $s_3 \not{E}_2 s_5$.
Hence P_2 is given by $P_2: \{s_1\}, \{s_2, s_5\}, \{s_3\}, \{s_4\}$.
$(v(s_2, x) = s_3) E_2 (v(s_5, x) = s_3)$ for $x = 0{,}1$.
Hence $s_2 E_3 s_5$ and $P\_2 = P\_3$.
Consequently, states s_2 and s_5 are equivalent.

b) States s_2 and s_5 are equivalent.

c) States s_2 and s_7 are equivalent; s_3 and s_4 are equivalent

Ex 7.5: (3)

a) s_1 and s_7 are equivalent; s_4 and s_5 are equivalent

b) (i) 0000
(ii) 0
(iii) 00

	v		w	
M:	0	1	0	1
s_1	s_4	s_1	1	0
s_2	s_1	s_2	1	0
s_3	s_6	s_1	1	0
s_4	s_3	s_4	0	0
s_6	s_2	s_1	1	0