Name: Student ID:

Midterm #1 (35% + 5% Bonus Points)

CS2336 Discrete Mathematics, Instructor: Cheng-Hsin Hsu

Department of Computing Science, National Tsing Hua University, Taiwan

3:30 - p.m., April 22th, 2012

This is a closed book test. Any academic dishonesty will automatically lead to zero point.

1) (5%) How many ways are there to place 13 balls of the same size in 6 distinct jars if
   a) all the balls are in red?
   b) each ball is in a different color?

   Answer:
   a) $\binom{13+6-1}{13} = \binom{18}{13} = 8568$
   b) 6^{13}

2) (5%) Answer the following questions

a) If statement q has the truth value 1, determine all truth value assignments for the primitive statements p, r, and s so that the truth value of the following statement is also 1: $q \rightarrow [(\neg p \wedge r) \vee \neg s] \vee [\neg s \rightarrow (\neg r \wedge q)]$.

b) Verify that $[p \rightarrow (q \rightarrow r)] \rightarrow [(p \rightarrow q) \rightarrow (p \rightarrow r)]$ is a tautology.

Answer:

a) When $s = 1$, the truth value of $[\neg s \rightarrow (\neg r \wedge q)]$ is 1.
When $s = 0$, the truth value of $[(\neg p \wedge r) \vee \neg s]$ is 1.
Furthermore, the statement q has the truth value 1,
so the statement $q \rightarrow [(\neg p \wedge r) \vee \neg s] \vee [\neg s \rightarrow (\neg r \wedge q)]$ is always true.
The truth values of p, r, and s could be assigned arbitrarily.

b)

p	q	r	$p \rightarrow (q \rightarrow r)$	$(p \rightarrow q) \rightarrow (p \rightarrow r)$	$[p \rightarrow (q \rightarrow r)] \rightarrow [(p \rightarrow q) \rightarrow (p \rightarrow r)]$
0	0	0	1	1	1
0	0	1	1	1	1
0	1	0	1	1	1
0	1	1	1	1	1
1	0	0	1	1	1
1	0	1	1	1	1
1	1	0	0	0	1
1	1	1	1	1	1

So $[p \rightarrow (q \rightarrow r)] \rightarrow [(p \rightarrow q) \rightarrow (p \rightarrow r)]$ is a tautology.

3) (5%) Define $p(x) : x^2 - 8x + 15 = 0$, $q(x) : x$ is odd, and $r(x) : x > 0$. Determine the truth or falsity of each of the statements. If a statement is false, you must give a valid counterexample to get the point.

a) $\forall x[p(x) \rightarrow q(x)]$
b) $\exists x[p(x) \rightarrow (q(x) \wedge r(x))]$
c) $\forall x[(p(x) \rightarrow q(x)) \rightarrow r(x)]$
d) $\forall x[\neg q(x) \rightarrow \neg p(x)]$
e) $\exists x[r(x) \rightarrow p(x)]$

Answer:

a) True.
b) True.
c) False. When $x = -1$.
d) True.
e) True.

4) (5%) Applying the laws of set theory, simplify each of the following:

a) $A \cap (B - A)$
b) $\overline{A} \cup \overline{B} \cup (A \cap B \cap \overline{C})$
c) $(A - B) \cup (A \cap B)$
d) $(A \cap B) \cup (A \cap B \cap \overline{C} \cap D) \cup (\overline{A} \cap B)$
e) $\overline{(A \cup B) \cap C}$

Answer:

a) $\emptyset$
b) $\overline{A} \cup \overline{B} \cup \overline{C}$
c) A
d) B
e) $(\overline{A} \cap \overline{B}) \cup \overline{C}$

5) (5%) Answer the following questions

a) How many permutations of the digits 0, 1, 2, $\ldots$, 9 either start with a 4 or end with a 9?

b) How many arrangements of the letters in CHEMIST have H before E, or E before T, or T before M? By before, we mean anywhere before, not just immediately before.

Answer:

a) $2 \cdot 9! - 8! = 685440$

b) $7! - 5 \cdot 6 \cdot 7 = 4830$

6) (5%) For $n \in \mathbb{Z}^+$, let $S(n)$ be the open statement:

$$\sum_{i=1}^{n} i = \frac{(n+0.5)^2}{2}. \tag{1}$$

a) Show that the truth of $S(k)$ implies the truth of $S(k+1)$ for all positive integer k.

b) Is $S(n)$ true for all $n \in \mathbb{Z}^+$?

Answer:

a) Assume $n = k$ is true. $S(k) = \sum_{i=1}^{k} i = \frac{(k+0.5)^2}{2}$.

Consider $n = k+1$.

$S(k+1) = \sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1) = \frac{(k+0.5)^2}{2} + (k+1) = \frac{k^2+3k+2.25}{2} = \frac{[(k+1)+0.5]^2}{2}$

So $S(k) \Rightarrow S(k+1)$ is truth.

b) When $n = 1$, $S(1) = \sum_{i=1}^{1} i = 1 \neq 1.125 = \frac{(1+0.5)^2}{2}$

$S(n)$ is not true for all $n \in \mathbb{Z}^+$.

7) (5%) Let $A = \{1, 2, 3, 4\}$ and $B = \{-1, -2, -3, -4, -5, -6\}$. Answer the following questions

a) How many functions are there from A to B?

b) How many functions are there from B to A?

c) How many one-to-one functions are there from A to B?

d) How many one-to-one functions are there from B to A?

e) How many onto functions are there from B to A?

Answer:

a) 6^4

b) 4^6

c) $\frac{6!}{2!} = 360$

d) 0

e) $4! \cdot S(6,4) = \binom{4}{4}4^6 - \binom{4}{3}3^6 + \binom{4}{2}2^6 - \binom{4}{1}1^6 = 1560$

8) (5%) For each of the following function $f : \mathbb{R} \to \mathbb{R}$, determine whether f is invertible and if so, give f^{-1}.

a) $f = \{(x, y)|2x + 3y = 7\}$
b) $f = \{(x, y)|ax + by = c, b \neq 0\}$
c) $f = \{(x, y)|y = x^4\}$
d) $f = \{(x, y)|y = x^3\}$
e) $f = \{(x, y)|y = x^4 + x\}$

Answer:

a) $f^{-1} = \{(x, y)|3x + 2y = 7\}$
b) If $a \neq 0$, $f^{-1} = \{(x, y)|bx + ay = c, b \neq 0, a \neq 0\}$
   If $a = 0$, f is not invertible.
c) f is not invertible.
d) $f^{-1} = \{(x, y)|y = x^{\frac{1}{3}}\}$
e) f is not invertible.