Name: Student ID:

Quiz #4 (4%)

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3:30 - 3:50 p.m., April 1st, 2013

This is a closed book test. Any academic dishonesty will automatically lead to zero point.

1) (1%) Give a recursive definition of the set of all
   a) negative even integers.
   b) perfect squares of all integers.

   Answer:
   a) Let A denote the set of all negative even integers.
      $-2 \in A$; and $\forall a \in A, a - 2 \in A$
   b) Let B denote the set of all perfect squares of all integers.
      $0 \in B$; and $\forall b \in B, b + 2\sqrt{b} + 1 \in B$

2) (1%)
   a) (0.5%) How many positive divisors are there for $n = 2^{14}3^{9}5^{8}7^{10}11^{3}13^{5}37^{10}$?
   b) (0.25%) How many of them are perfect cubes?
   c) (0.25%) How many of them are divisible of 1,166,400,000?

   Answer:
   a) $15 \times 10 \times 9 \times 11 \times 4 \times 6 \times 11 = 3920400$
   b) $5 \times 4 \times 3 \times 4 \times 2 \times 2 \times 4 = 3840$
   c) $6 \times 4 \times 4 \times 11 \times 4 \times 6 \times 11 = 278784$

3) (1%) Prove the following equations for $n \geq 1$ using mathematical induction.

a) $\sum_{i=1}^{n}(i)(i!) = (n+1)! - 1$.

b) $\sum_{i=1}^{n}(2^i)i - 2 = (n-1)2^{n+1}$.

Answer:

a) (1) $n = 1$, $\sum_{i=1}^{1}(i)(i!) = 1 = (1+1)! - 1$ is true.

(2) Assume $n = k$ is true: $\sum_{i=1}^{k}(i)(i!) = (k+1)! - 1$

Consider $n = k+1$:

$\sum_{i=1}^{k+1}(i)(i!) = \sum_{i=1}^{k}(i)(i!) + (k+1)(k+1)! = (k+1)! - 1 + (k+1)(k+1)! = (k+2)! - 1$

From (1)(2), $\sum_{i=1}^{n}(i)(i!) = (n+1)! - 1$ is true by the Principle of Mathematical Induction.

b) (1) $n = 1$, $\sum_{i=1}^{1}(2^i)i - 2 = 0 = (1-1)2^{1+1}$ is true.

(2) Assume $n = k$ is true: $\sum_{i=1}^{k}(2^i)i - 2 = (k-1)2^{k+1}$

Consider $n = k+1$:

$\sum_{i=1}^{k+1}(2^i)i - 2 = \sum_{i=1}^{k}(2^i)i - 2 + 2^{k+1}(k+1) = (k-1)2^{k+1} + 2^{k+1}(k+1) = [(k+1)-1]2^{(k+1)+1}$

From (1)(2), $\sum_{i=1}^{n}(2^i)i - 2 = (n-1)2^{n+1}$ is true by the Principle of Mathematical Induction.

4) (1%) Define the set $X \subseteq \mathbb{Z}^+$ as follows: (i) $3 \in X$ and (ii) if $a, b \in X$, then $a + b \in X$. Prove that X is the set of all positive integers divisible by 3.

Answer:

Let $Y = \{3k | k \in \mathbb{Z}^+\}$, the set of all positive integers divisible by 3.

In order to show that $X = Y$ we shall verify that $X \subseteq Y$ and $Y \subseteq X$.

- ($X \subseteq Y$): By part (1) of the recursive definition of X we have 3 in X. And since $3 = 3 \cdot 1$, it follows that 3 is in Y. Turning to part (2) of this recursive definition suppose that for $x, y \in X$ we also have $x, y \in Y$. Now $x + y \in X$ by the definition and we need to show that $x + y \in Y$. This follows because $x, y \in Y \Rightarrow x = 3m, y = 3n$ for some $m, n \in \mathbb{Z}^+ \Rightarrow x + y = 3m + 3n = 3(m + n)$, with $m + n \in \mathbb{Z}^+ \Rightarrow x + y \in Y$. Therefore every positive integer that results from either part (1) or part (2) of the recursive definition of X is an element in Y, and, consequently, $X \subseteq Y$.

- ($Y \subseteq X$): In order to establish this inclusion we need to show that every positive integer multiple of 3 is in X. This will be accomplished by the Principle of Mathematical Induction. Start with the open statement $S(n) : 3n$ is an element in X, which is defined for the universe $\mathbb{Z}^+$. The basis step - that is, $S(1)$ - is true because $3 \cdot 1 = 3$ is in X by part (1) of the recursive definition of X.

  For the inductive step of this proof we assume the truth of $S(k)$ for some $k (\geq 1)$ and consider what happens at $n = k + 1$.

  From the inductive hypothesis $S(k)$ we know that $3k$ is in X. Then from part (2) of the recursive definition of X we fond that $3(k+1) = 3k+3 \in X$, because $3k, 3 \in X$. Hence $S(k) \Rightarrow S(k + 1)$.

  So by the Principle of Mathematical Induction it follows that $S(n)$ is ture for all $n \in \mathbb{Z}^+$ - and, consequently, $Y \subseteq X$.

With $X \subseteq Y$ and $Y \subseteq X$ it follows that $X = Y$.