Name: Student ID:

Quiz #9 (4%)

CS2336 Discrete Mathematics, Instructor: Cheng-Hsin Hsu

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2:20 - 2:40 p.m., May 30th, 2013

This is a closed book test. Any academic dishonesty will automatically lead to zero point.

1) (1%) If $a_n, n \geq 0$, is the unique solution of the recurrence relation $a_{n+1} - da_n = 0$, and $a_3 = 153/49, a5 = 1377/2401$, what is d?

   Answer:

   $a_5 = d \cdot a_4 = d^2 \cdot a_3$

   $\Rightarrow d^2 = \frac{a_5}{a_3} = \frac{9}{49}$

   $\Rightarrow d = \pm\frac{3}{7}$

2) (1%) If $a_0 = 0$, $a_1 = 1$, $a_2 = 4$, and $a_3 = 37$ satisfy the recurrence relation $a_{n+2} + ba_{n+1} + ca_n = 0$, where $n \geq 0$ and b, c are constants, determine b, c, and solve for a_n.

   Answer:

   If $n = 0$: $ca_0 + ba_1 + a_2 = 0 \Rightarrow b + 4 = 0 \Rightarrow b = -4$.

   If $n = 1$: $ca_1 + ba_2 + a_3 = 0 \Rightarrow c + 4b + 37 = 0 \Rightarrow c = -21$.

   $\Rightarrow a_{n+2} - 4a_{n+1} - 21 = 0$

   $\Rightarrow x^2 - 4x - 21 = 0 \Rightarrow x = 7, -3$

   $\Rightarrow a_n = A \cdot 7^n + B \cdot (-3)^n \Rightarrow \begin{cases} a_0 = A + B = 0 \\ a_1 = 7A - 3B = 1 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{10} \\ B = -\frac{1}{10} \end{cases}$

   $\Rightarrow a_n = \frac{7^n}{10} - \frac{(-3)^n}{10}$

3) (1%) Solve the following recurrence relations: (a) $a_{n+2} + 3a_{n+1} + 2a_n = 3^n, n \geq 0, a_0 = 0, a_1 = 1$, (b)$a_{n+2} + 4_{n+1} + 4a_n = 7, n \geq 0, a_0 = 1, a_1 = 2$.

Answer:

a) $x^2 + 3x + 2 = 0 \Rightarrow x = -2, -1 \Rightarrow a_n^{(h)} = A(-2)^n + B(-1)^n$

And $a_n^{(p)} = C3^n \Rightarrow C3^{n+2} + 3C3^{n+1} + 2C3^n = 3^n \Rightarrow C = \frac{1}{20}$

Then $a_n = a_n^{(h)} + a_n^{(p)} = A(-2)^n + B(-1)^n + \frac{1}{20}3^n$

$$\Rightarrow \begin{cases} a_0 = A + B + \frac{1}{20} = 0 \\ a_1 = -2A - B + \frac{3}{20} = 1 \end{cases} \Rightarrow \begin{cases} A = \frac{-4}{5} \\ B = \frac{3}{4} \end{cases}$$

So $a_n = \frac{-4}{5}(-2)^n + \frac{3}{4}(-1)^n + \frac{1}{20}3^n$

b) $x^2 + 4x + 4 = 0 \Rightarrow x = -2, -2 \Rightarrow a_n^{(h)} = A(-2)^n + Bn(-2)^n$

And $a_n^{(p)} = C \Rightarrow C + 4C + 4C = 7 \Rightarrow C = \frac{7}{9}$

Then $a_n = a_n^{(h)} + a_n^{(p)} = A(-2)^n + Bn(-2)^n + \frac{7}{9}$

$$\Rightarrow \begin{cases} a_0 = A + \frac{7}{9} = 1 \\ a_1 = -2A - 2B + \frac{7}{9} = 2 \end{cases} \Rightarrow \begin{cases} A = \frac{2}{9} \\ B = \frac{-5}{6} \end{cases}$$

So $a_n = \frac{2}{9}(-2)^n + \frac{-5}{6}n(-2)^n + \frac{7}{9}$

4) (1%) Solve the following recurrence relation by the method of generating functions: $a_{n+2} - 2a_{n+1} + a_n = 2^n$, $n \geq 0, a_0 = 1, a_1 = 2$.

Answer:

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

$\Rightarrow \sum_{n=0}^{\infty}(a_{n+2})x^{n+2} - \sum_{n=0}^{\infty}(2a_{n+1})x^{n+2} + \sum_{n=0}^{\infty}(a_n)x^{n+2} = \sum_{n=0}^{\infty}(2^n)x^{n+2}$

$\Rightarrow \sum_{n=0}^{\infty}(a_{n+2})x^{n+2} - 2x\sum_{n=0}^{\infty}(a_{n+1})x^{n+1} + x^2\sum_{n=0}^{\infty}(a_n)x^n = x^2\sum_{n=0}^{\infty}(2^n)x^n$

$\Rightarrow (f(x) - 1 - 2x) - 2x(f(x) - 1) + x^2(f(x)) = x^2\frac{1}{1-2x}$

$\Rightarrow f(x) = \frac{1}{1-2x} = \sum_{n=0}^{\infty}(2^n)x^n = \sum_{n=0}^{\infty}(a_n)x^n$

So $a_n = 2^n$.