CS 2336: Discrete Mathematics

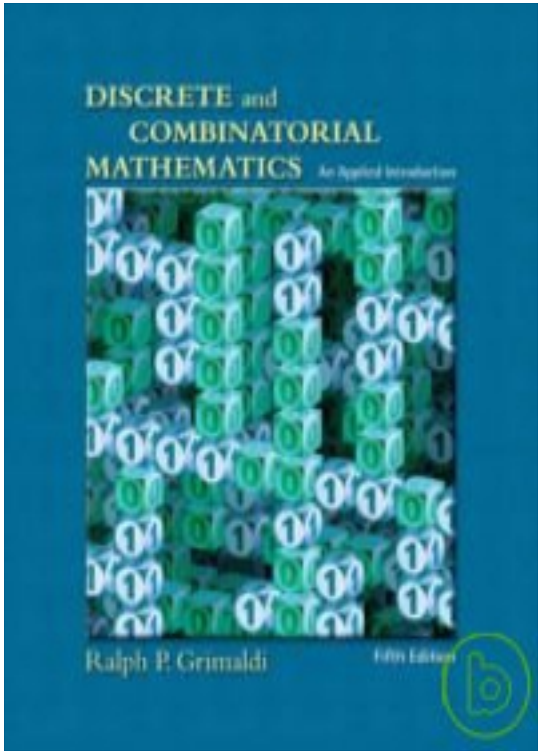
Instructor: Cheng-Hsin Hsu

Course Information

- **TextBook**: Discrete and Combinatorial Mathematics, R. Grimaldi, 5th Ed., Addison Wesley
- **Lecture**: Mondays 3:20-5:10 p.m.

  and Thursdays 2:10-3:00 p.m.
- **Location**: 台達館 109
- **Office Hour:** Thursday 3:00 – 4:00 p.m.
- **Course Website**:

  http://nmsl.cs.nthu.edu.tw/index.php/courses
- **TA**: Liao, Chen-Chih (u9762245@oz.nthu.edu.tw)



Topics We Plan to Cover

1. Fundamental Principles of Counting (Chapter 1 in textbook)
2. Fundamental of Logic (Chapter 2)
3. Set Theory (Chapter 3)
4. Properties of Integers: Mathematical Induction (Chapter 4)
5. Relations and Functions (Chapter 5)
6. Relations: The Second Time Around (Chapter 7)
7. The Principle of Inclusion and Exclusion (Chapter 8)
8. Generating Functions (Chapter 9)
9. Recurrence Relations (Chapter 10)
10. An Introduction to Graph Theory (Chapter 11)
11. Trees (Chapter 12)
12. Optimization and Matching (Chapter 13)

Fundamentals

Enumeration

Graph Theory

Not Covered: Applied Algebra

Grading Policy

- **Quizzes (48%):** One for each chapter
  - Given on Mondays, either at the beginning or end of the lecture
  - Sample questions will be given as homework, which is not collected nor graded
  - No makeup quizzes, unless an email requesting for a leave is sent to and approved by the instructor before each quiz
- **Midterm Exam (35%)**: 2-hr exam
- **Final Exam (35%)**: 2-hr exam

Tentative Schedule

Week	Monday	Thursday	Quiz Solutions
1: Feb 18	Introduction	Ch. 1 Fundamental Principles of Counting	
2: Feb 25	No lecture: conference travel	Holiday	
3: Mar 4	Ch. 1 Fundamental Principles of Counting	Ch. 2 Fundamental of Logic	
4: Mar 11	Ch. 2 Fundamental of Logic	Ch. 3 Set Theory	Quiz 1 (Ch. 1)
5: Mar 18	Ch. 3 Set Theory	Ch. 4 Properties of Integers: Mathematical Induction	Quiz 2 (Ch. 2)
6: Mar 25	Ch. 4 Properties of Integers: Mathematical Induction	Ch. 5 Relations and Functions	Quiz 3 (Ch. 3)
7: Apr 1	Ch. 5 Relations and Functions	Holiday	Quiz 4 (Ch. 4)
8: Apr 8	Ch. 7 Relations: The Second Time Around	Ch. 7 Relations: The Second Time Around	Quiz 5 (Ch. 5)
9: Apr 15	Mid Term Exam (Ch. 1 - Ch. 5)	Ch. 7 Relations: The Second Time Around	
10: Apr 22	Ch. 8 The Principle of Inclusion and Exclusion	Ch. 8 The Principle of Inclusion and Exclusion	Quiz 6 (Ch. 7)
11: Apr 29	Ch. 9 Generating Functions	Ch. 9 Generating Functions	Quiz 7 (Ch. 8)
12: May 6	Ch. 10 Recurrence Relations	Ch. 10 Recurrence Relations	Quiz 8 (Ch. 9)
13: May 13	Ch. 11 An Introduction to Graph Theory	Ch. 11 An Introduction to Graph Theory	Quiz 9 (Ch. 10)
14: May 20	Ch. 11 An Introduction to Graph Theory	Ch. 12 Trees	Quiz 10 (Ch. 11)
15: May 27	Ch. 12 Trees	Ch. 13 Optimization and Matching	Quiz 11 (Ch. 12)
16: Jun 3	Ch. 13 Optimization and Matching	Practice Final Exam	Quiz 12 (Ch. 13)
17: Jun 10	Final Exam (Ch. 7 - 13)		
18: Jun 17			

What is Discrete Mathematics?

- Covers various kinds of topics
  - Logics
  - Combinatorial
  - Algorithms
  - Graph Theory
  - Number Theory
- Discrete ← something you can count

What is a Proof?

- Vaguely speaking:
  - To convince someone that something is true
- Mathematical proof:
  - To show if some axioms are true, then some statement is also true
- What we will do in this course is somewhere between these two definitions
  - You need to get the main idea
  - Details are not important

Examples of Proofs

- Prove that $\sqrt{2}$ is irrational
  - Definition of rational?
  - Proof by contradiction ← if we write $\sqrt{2}$ as p/q, p and q must be both even
- Prove that there are infinite prime numbers
  - How to create a new prime number given a bunch of prime numbers?
  - Given a set of prime number $\mathbf{P}$, $\prod_{p \in \mathbf{P}} +1$ is either a prime number or has a prime factor that is not in $\mathbf{P}$

How Math Abstraction Helps?

- Consider two problems
  - Pick 2 out of 11 students, how many different ways can we choose?
  - How many bowling pins do we have if there are 10 rows?
- Connect them using triangular number
  - $T_n = 1 + 2 + \cdots + n$
- Prove $T_n = \frac{n(n+1)}{2}$
  - Geometrical
  - Algebraic

Mathematical Induction

- We cut a cake *n* times and there are no three cuts intersecting at a point. How many pieces of cake do we get?
  - Observe that $P_n = T_n + 1$ ← does this always hold?
  - Another observation: for *k*-th cut, we create *k* additional pieces
  - Prove it by induction
- Induction is useful even if we don't really understand why the proposition holds

Questions?

Questions so far?

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Chapter 1

Fundamental Principles of Counting

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Rule of Sum

- Q: There are 6, 8, and 12 introductory books for C++, Java, and Perl, respectively. If Joe wants to learn a first language, how many choices (of books) does he have?
- Q: Say Joe has two friends, who own 3 and 5 of these books. How many unique books can Joe borrow from his friends?
- Rule: If task 1 can be performed in m ways, task 2 can be performed in n ways, and tasks 1 and cannot be performed concurrently, then there are $m+n$ ways to perform either task 1 or 2.

Rule of Product

- Q: There is a singer try-out with 4 men and 6 women, how many different way to form a couple of singers?
- Q: How many different license plates of two letters followed by four digits can we produce with/without repetition
- Rule: If task 1 can be performed in m ways, task 2 can be performed in n ways, there are mn ways to sequentially perform tasks 1 and 2.

Combinations of Two Rules

- Q: A cafeteria offers 6 kinds of muffins, 8 kinds of sandwiches, and 3 different beverages, and each combo consists of a muffin (or sandwiches) and a beverage. How many different combos does this cafeteria offer?

Permutation

- Q: From a class of 10 students, choose and seat 5 of them in a line for a picture. How many linear arrangement are possible?
  - Write the solution using *n* factorial
- Definition: Given a collection of *n* distinct object, any arrangement of these objects is called a permutation of the collection
- For *n* distinct objects and an integer $1 \leqslant r \leqslant n$, per the rule of product, the number of permutation of size *r* of *n* object is: $P(n,r) = \frac{n!}{(n-r)!}$

Repeated Permutation

- Q: How many permutations of the letters in "COMPUTER" if
  - Repetitions are not allowed
  - Repetitions are allowed
- Q: How many permutations of the letters in
  - BALL
  - DATABASES
- General Result: If there are n objects with n_1 (indistinguishable) type 1 objects,…, n_r type r objects, there are $\frac{n!}{n_1!n_2!\cdots n_r!}$ arrangement of the given n objects

Nonlinear Permutation

- Q: If six students are seated at a round table, how many different arrangements are possible?
  - Assume that arrangements are considered the same when once can be derived by rotating the other
- Q: How many ways we can arrange 3 males and 3 female around a table so that the sexes alternate?

Combinations

- Q: How many possible permutations of 3 from 52 cards? What if the cards are considered as unordered?
- General rule: For *n* distinct object, the number of combinations of *r* object, where $0 \leqslant r \leqslant n$, is:

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$$

- Q: How many different ways a student can answer 7 out of 10 questions in a test?
- Q: What if the student needs to pick 3 questions from the first 5 questions and 4 questions from the rest?

Permutation and/or Combination

- Q: How many arrangements of the letters in TALLAHASSEE with no adjacent A's?
  - Requires both permutation and combination
- Q: The coach wants to form four teams of 9 students each from a class of 36 students. Call the teams A, B, C, and D, how many different ways can the coach form the teams?
  - Solve it using combinations
  - Solve it using permutations

Summation

- We use Greek symbol sigma to represent summations
  - $\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_n$
  - i is the index
  - m is the lower limit
  - n is the upper limit
- Example:

$$\sum_{i=2}^{3} \binom{5}{7-j}\binom{5}{j} = \binom{5}{5}\binom{5}{2} + \binom{5}{4}\binom{5}{3}$$

Another Example

- Q: Let n be a positive integer, there are 3^n strings of a alphabet consisting of symbols 0, 1, and 2. Define weight(x)=$x_1+x_2+\ldots+x_n$, for $x=x_1x_2\ldots x_n$. For $n=10$, how many strings with even weights?

Theorems

- Lemma: Prove that for two integers *n* and *r*, $n \geqslant r \geqslant 0$, we have $\binom{n}{r} = \binom{n}{n-r}$

- Theorem (Binomial): Prove that

$$(x+y)^n = \binom{n}{0}x^0y^n + \binom{n}{1}x^1y^{n-1} + \cdots + \binom{n}{n}x^ny^0 = \sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}$$

  - $\binom{n}{k}$ is called binomial coefficient

- Corollary
  - $2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$
  - $0 = \binom{n}{0} - \binom{n}{1} + \cdots + (-1)^n\binom{n}{n}$

Theorems (cont.)

- Theorem (Multinomial): Prove that the coefficient of $x_1^{n_1} x_2^{n_2} \dots x_t^{n_t}$ in the expansion of $(x_1+x_2+\dots+x_t)^n$ is

$$\frac{n!}{n_1! n_2! \dots n_t!}$$

- $\binom{n}{n_1, n_2, \dots, n_t}$ is called multinomial coefficient

- Example: Determine the coefficient of $a^2 b^3 c^2 d^5$ in the expansion of $(a + 2b - 3c + 2d + 5)^{16}$

Comb. with Repetition

- Q: Seven students stop at a fast food restaurant where each of them can order a burger, a hot dog, a taco, or a sandwich. The restaurant only cares about how many burgers, hot dogs, tacos, and sandwiches do the students order. What is the number of possible solution?
- General Rule: The number of combinations of, with repetition, *r* objects from *n* distinct objects is:

$$\frac{(n+r-1)!}{r!(n-1)!} = \binom{n+r-1}{r}$$

Examples

- Q: A donut shop offer 20 kinds of donuts. Assuming that there are plenty of donuts of each kind, how many ways for a kid to buy a dozen donuts?
- Q: A father distributes \$1000 among 4 kids at a step of \$100
  - How many different ways to distribute \$1000 if some kids may get nothing?
  - How many ways to distribute \$1000 if each kid is guaranteed to have at least \$100?
- Q: How many solutions does $x_1 + x_2 + x_3 + x_4 = 7$ have, for positive $x_{1,}$ x_2, x_3, *and* x_4?

Equivalence

- The number of selections, with repetition, of size r from a collection of size n
- The number of ways r identical objects can be distributed among n distinct containers
- The number of integer solutions of the equation

$$x_1 + x_2 + \cdots + x_n = r, \quad x_i \geqslant 0, 1 \leqslant i \leqslant n$$

Examples

- Q: How many nonnegative solutions does the inequality $x_1 + x_2 + \cdots + x_6 < 10$ have?

- Q: How many times the print statement is executed?

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for i = 1 to 20
    for j = 1 to i
        for k = 1 to j
            print (i+j+k);
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Take-home Exercise

- Exercise 1.1 and 1.2: 15, 22, 28, 32, 33
- Exercise 1.3: 13, 16, 25, 29, 34
- Exercise 1.4: 7, 17, 24, 26, 28