Department of Computer Science
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CS 2336: Discrete Mathematics

Chapter 4

Properties of the Integers: Mathematical Induction

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Outline

4.1 The Well-ordering Principle: Mathematical Induction

4.2 Recursive Definitions

4.3 The Division Algorithm: Prime Numbers

4.4 The Greatest Common Divisor: The Euclidean Algorithm

4.5 The Fundamental Theorem of Arithmetic

Well-Ordering Principle

- What makes ***Z*** different from ***Q*** and ***R***?
- Observation: $\mathbb{Z}^+ = \{x \in \mathbb{Z} | x > 0\} = \{x \in \mathbb{Z} | x \geqslant 1\}$
  - but: $\mathbb{Q}^+ = \{x \in \mathbb{Q} | x > 0\}, \ \mathbb{R}^+ = \{x \in \mathbb{R} | x > 0\}$
- Every nonempty subset ***X*** of $\boldsymbol{Z}^+$ contains a least (smallest) element
  - Why it's not true for $\boldsymbol{Q}^+$ and $\boldsymbol{R}^+$?
- This is called the Well-Ordering Principle
  - We say $\boldsymbol{Z}^+$ is well-ordered

Principle of Mathematical Induction

- Let $S(n)$ denote an open statement that involves the positive integer variable n
  - $S(1)$ is true and ← basis step
  - When $S(k)$ is true then $S(k+1)$ is true ← inductive step

  Then $S(n)$ is true for all n in $\boldsymbol{Z}^+$
- Extension:
  - May use $S(k_0)$ instead of $S(1)$ as the basis step
  - Can expand $\boldsymbol{Z}^+$ into

    $\{x|x \in \mathbb{Z}, x > n_0\}$, where $n_0 < 0$ is a finite number

Examples

- Ex 4.1: Prove $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \ \forall n \in \mathbb{Z}^+$
- Ex 4.4: Prove $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \ \forall n \in \mathbb{Z}^+$
- Ex 4.6: Check if the inductive step of the following (invalid) theorem works?

$$S(n) : \sum_{i=1}^{n} i = \frac{n^2+n+2}{2} \ \forall n \in \mathbb{Z}^+$$

- Ex 4.13: Prove that any interger larger than or equal to 14 can be written as a sum of only 3's and 8's.

Alternative Form

- Let $S(n)$ denote an open statement that involves the positive integer variable n, let $n_0 <= n_1$ be two positive integers
  - $S(n_0)$, $S(n_0+1)$, …, $S(n_1-1)$, $S(n_1)$ are true and
  - When $S(n_0)$ … $S(k)$ are true, where $k >= n_1$ then $S(k+1)$ is true

Then $S(n)$ is true for all $n >= n_0$

Example

- Ex 4.14 (alternative proof): It is possible to write *14*, *15*, *16* using only *3*'s and *8*'s:
  - $14=3+3+8$
  - $15=3+3+3+3+3$
  - $16=8+8$

  Prove

  $S(n)$: n can be written as a sum of *3*'s and *8*'s

  is true for all positive integer $n >= 14$

Outline

Explicit Formula

- A sequence of integers may or may not be written in explicit formula (depending on if you can observer a pattern!)
  - 0, 2, 4, 8, 10, 12,…
  - 1, 2, 3, 6, 11, 20, 37,…
- For those sequences that do not have explicit formulas, we may define it recursively:
  - E.g., $a_0=1$, $a_1=2$, $a_2=3$, and $a_n = a_{n-1}+a_{n-2}+a_{n-3}$
- Not necessary for sequence, but also for general mathematical concepts
  - e.g., conjunction of multiple statements

Recursive Definition

- Ex: 4.17 Considers sets $A_1, A_2, \ldots, A_n, A_{n+1},$ where $A_i \subseteq \mathscr{U}$ we define their union recursively as
  - The union of A_1, A_2 is $A_1 \cup A_2$ ← base definition
  - The union of $A_1, A_2, \ldots, A_{n+1}$ for $n \geqslant 2$, is given by

$$A_1 \cup A_2 \cup \cdots \cup A_n \cup A_{n+1} = (A_1 \cup A_2 \cup \cdots \cup A_n) \cup A_{n+1}$$

←recursive process

- Then we have

$$(A_1 \cup A_2 \cup \cdots \cup A_r) \cup (A_{r+1} \cup \cdots \cup A_n) = A_1 \cup A_2 \cup \cdots \cup A_r \cup A_{r+1} \cup \cdots \cup A_n$$

if $n, r \in \mathbb{Z}^+$ where $n \geqslant 3, 1 \leqslant r < n$

← prove this using induction

Harmonic Numbers

- Define Harmonic numbers *H* as
  - $H_1=1$
  - $H_{n+1}=H_n+1/(n+1)$ for $n >= 1$
- Prove $\sum_{j=1}^{n} H_j = (n+1)H_n - n, \ \forall n \in \mathbb{Z}^+$
- Another example of recursive definition: factorial
  - $0! = 1$
  - $(n+1)! = (n+1)\ n!$, for all $n >= 0$
- Define even number as a sequence $b_0, b_1, b_2, \ldots$ using recursive definition

Fibonacci Numbers

- Define Fibonacii numbers F as
  - $F_0=0$, $F_1=1$
  - $F_n = F_{n-1}+F_{n-2}$ for $n >= 2$
- Ex 4.19: Prove $\sum_{i=0}^{k} F_i^2 = F_k \times F_{k+1}, \forall n \in \mathbb{Z}^+$

Lucas Numbers

- Define Lucas numbers L as
  - $L_0=2$, $L_1=1$
  - $L_n=L_{n-1}+L_{n-2}$, for $n>=2$

- Ex 4.20: Prove: $L_n = F_{n-1} + F_{n+1}, \ \forall n \in \mathbb{Z}^+$

Table 4.2

n	0	1	2	3	4	5	6	7
L_n	2	1	3	4	7	11	18	29

Recursively Defined Set

- Start from an initial set of element with one/ multiple rules to create new elements based on the known element
  - All the elements in the recursively defined set either belong to the initial set, or were created by the rules

- Example 4.22: Define the set X recursively by: (i) 1 is in X, and (ii) for each a in X, $a+2$ is also in X.

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Definition 4.1

- For $a, b \in \mathbb{Z}$ and $b \neq 0$, we say that *b* divides *a*, or $b|a$, if there is an integer *n* such that *a*=*bn*. In this case, *b* is a divisor of *a* and *a* is a multiple of *b*.

- Properties for $a, b, c \in \mathbb{Z}$
  - $1|a$ and $a|0$
  - $[(a|b) \wedge (b|a)] \Rightarrow a = \pm b$
  - $[(a|b) \wedge (b|c)] \Rightarrow a|c$
  - $a|b \Rightarrow a|bx \ \forall x \in \mathbb{Z}$
  - If *x*=*y*+*z* and *a* divides two out of three integers, it divides the last one as well
  - $[(a|b) \wedge (a|c)] \Rightarrow a(bx + cy)$
  - $a|(x_1x_1 + \cdots + c_nx_n)$ if $a|c_i, \ \forall 1 \leqslant i \leqslant n$

Examples

- Ex 4.23: Do there exist integers x, y, and z, so that $6x+9y+15z=107$?

- Ex 4.24: a, b are two integers and $2a+3b$ is a multiple of 17. Show that 17 divides $9a+5b$.

Primes and Composite

- Primes are integers ($n>1$) with exactly two positive divisors
- All other integers ($n>1$) are called composite

- Lemma: If $n \in \mathbb{Z}^+$ is composite, then there is a prime p such that $p|n$ ← Well-Ordering Principle
- Theorem: There are infinitely many primes. How to prove it? ← By contradiction

The Division Algorithm

- For any $a, b \in \mathbb{Z}, b > 0$, there exist unique $q, r \in \mathbb{Z}$ with $a = qb + r,\ 0 \leqslant r < b$
  - q is called quotient
  - r is called remainder
  - a is called dividend
  - b is called divisor
- Ex 4.25: Find the q and r for the following a and b
  - $a = 170, b = 11$
  - $a = -45, b = 8$

Integers in Bases Other than 10

- Ex 4.27: Write *6137* in the octal system (base *8*). In other words, finds $r_0, r_1, \ldots, r_k$ so that $(6137)_{10}=(r_k\ldots r_2 r_1 r_0)_8$.
- Ex 4.28: write *3387* into binary (base *2*) and hexadecimal (base *16*).

	Remainders
8 | 6137	
8 | 767	1(r_0)
8 | 95	7(r_1)
8 | 11	7(r_2)
8 | 1	3(r_3)
0	1(r_4)

	Remainders	
16 | 13,874,945		
16 | 867,184	1	(r_0)
16 | 54,199	0	(r_1)
16 | 3,387	7	(r_2)
16 | 211	11 (= B)	(r_3)
16 | 13	3	(r_4)
0	13 (= D)	(r_5)

Negative Integers

- Question: How to represent negative integers x in binary?
  - One's complement: write $|x|$ in binary, and replace each 0 (1) with 1(0)
  - Two's complement: add 1 to one's complement
- Ex 4.29: Write -5 as two's complement in 4- and 8-bit integers
- Ex 4.30: Perform the subtraction 33-5 in base 2 8-bit integers ← observe the overflow, in this case we discard the left-most bit

Outline

Common Divisor

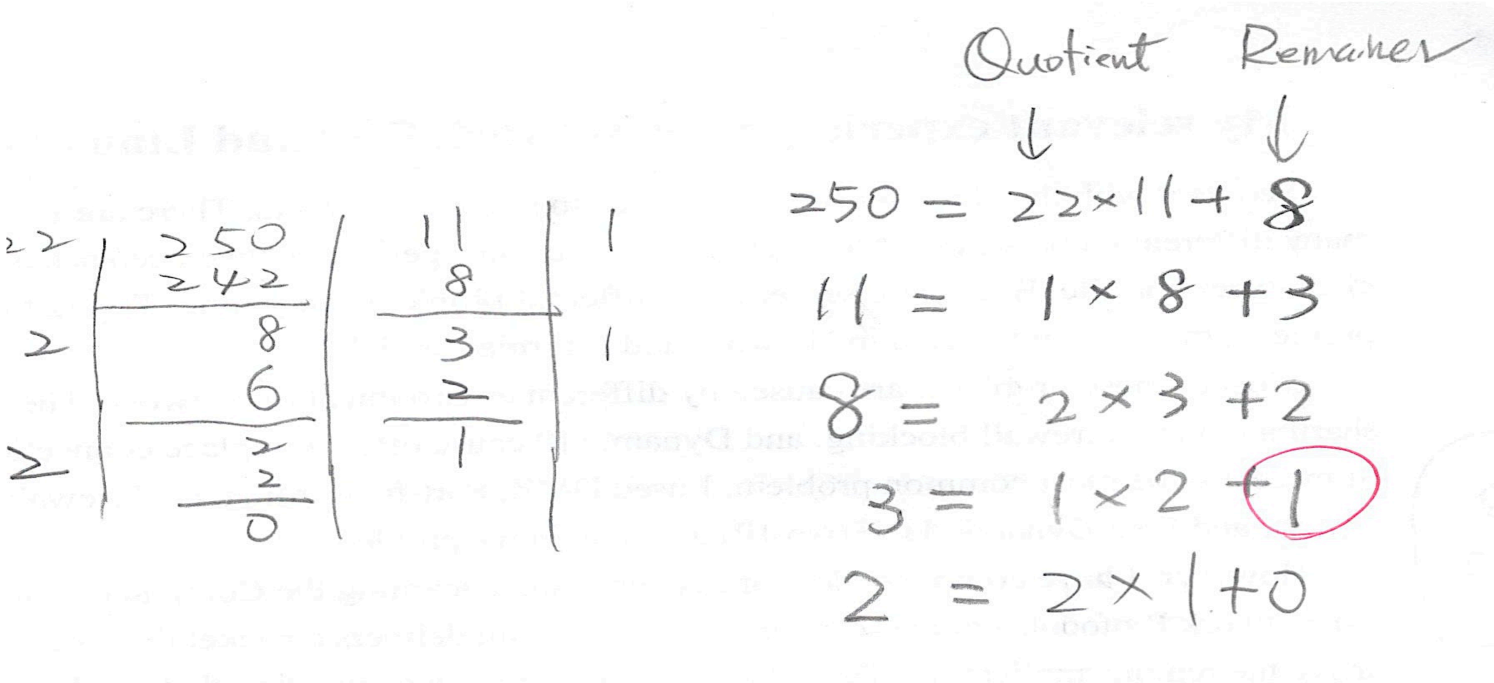
- For $a, b \in \mathbb{Z}$, $c > 0$ is a common divisor of *a* and *b* if $c|a$ and $c|b$
- Let $a, b \in \mathbb{Z}$, where $a \neq 0$ or $b \neq 0$. Then $c \in \mathbb{Z}^+$ is a greatest common divisor of *a* and *b* if
  - $c|a, c|b$
  - For any common divisor *d* of *a* and *b*, we know $d|c$
- Theorem 4.6 For all $a, b \in \mathbb{Z}^+$, there exists a unique greatest common divisor of *a* and *b*, written as *gcd*(*a*,*b*) ← Well-Ordering Principle, *gcd*(*a*,*b*) is actually the smallest positive integer that can is a linear combination of *a* and *b*

A Few Facts on GCD

- $gcd(a,b) = gcd(b,a)$
- $gcd(a,0)=|a|$, for any nonzero a
- $gcd(-a,b)=gcd(a,-b)=gcd(-a,-b)=gcd(a,b)$
- gcd(0,0) *is undefined.*

- Integer *a* and *b* are relatively prime if $gcd(a,b)=1$
  - If there exist integers *x* and *y*, so that $ax+by=1$

Euclidean Algorithm



- Then, r_n , the last nonzero remainder, equals *gcd*(*a*,*b*)
- Ex 4.34: Find the *gcd*(*250*,*11*)?

Examples

- Ex 4.35: Prove that 8n+3 and 5n+2 are relative prime
- Ex 4.36: Realize the Euclidean algorithm

```
$ cat GCD.java
public class GCD{
    public static void main(String[] args) {
        // a, b are positive integers
        int a = 120, b = 32;
        int r = a % b;
        int d = b;
        while (r > 0) {
            int c = d;
            d= r;
            r = c % d;
        }
        // gcd(amb) is d the last nonzero remainder
        System.out.println("gcd(" + a + ", " + b + ") = " + d);
    }
}
$ java GCD
gcd(120, 32) = 8
```

Diophanine Equation

- For positive integers a, b, c, the Diophantine equation $ax+by=c$ has an integer solution $x=x_0$, $y=y_0$ if $gcd(a,b)$ divides c

- Ex 4.38: Brian can debug a Java program in 6 mins and a C++ program in 10 mines. If he continuously works for 104 mins and doesn't waste any time, how many programs can he debug in each languages?
  - Basically find integers x and y so that $6x+10y=104$

Common Multiple

- Let $a, b \in \mathbb{Z}^+$. c is a common multiple of a and b. c is the least common multiple if it is the smallest positive common multiple of a, b, we write c=$lcm(a,b)$
- If $a, b \in \mathbb{Z}^+$ and $c = \text{lcm}(a, b)$. For any d that is a common multiple of a and b, we know $c|d$
- Thm 4.40: For all $a, b \in \mathbb{Z}^+$, ab=lcm(a,b)gcd(a,b)

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Fundamental Theorem of Arithmetic

- Lem 4.2: If $a, b \in \mathbb{Z}^+$ and *p* is a prime, then $p|ab \Rightarrow p|a \text{ or } p|b$
- Lem 4.3: Generalize Lem 4.2 to *n* positive integers
- Thm 4.11: Integer *n>1* can be written as a (unique) product of primes

- Ex 4.42: What is the prime factorization of 980,220?
- Ex 4.43: Prove that *17|n* given

$$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot n = 21 \cdot 20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14$$

Examples

- Ex 4.44: Count the number of positive divisors of 360.
- Ex 4.45: Let $m = p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}, n = p_1^{f_1} p_2^{f_2} \cdots p_t^{f_t},$ with $e_i, f_i \geqslant 0, \forall e_i, f_i$ we have

$$gcd(m, n) = \prod_{i=1}^{t} p_i^{a_i}, \text{ and } lcm(m, n) = \prod_{i=1}^{t} p_i^{b_i},$$

$$\text{where } a_i = \min(e_i, f_i), b_i = \max(e_i, f_i)$$

  - Find the gcd and lcm of $491891400 = 2^3 3^3 5^2 7^2 11^1 13^2$ and $1138845708 = 2^2 3^2 7^1 11^2 13^3 17^1$

Take-home Exercises

- Exercise 4.1: 2, 8, 16, 19, 26
- Exercise 4.2: 1, 8, 10, 12, 16
- Exercise 4.3: 7, 15, 20, 22, 28
- Exercise 4.4: 1, 2, 7, 14, 19
- Exercise 4.5: 1, 2, 8, 24, 25