Solution

Ex 5.1: 1, 3, 6, 8, 12

Ex 5.2: 4, 8, 15, 20, 27

Ex 5.3: 1, 4, 8, 12, 16

Ex 5.4: 1, 2, 5, 8, 12

Ex 5.5: 2, 6, 13, 14, 20

Ex 5.6: 7, 10, 16, 17, 22

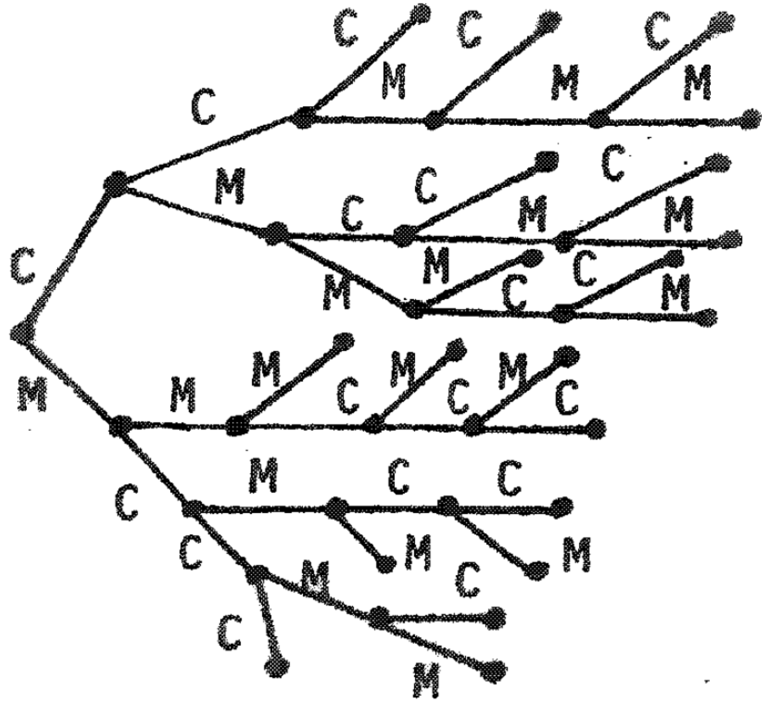
Ex 5.1: (1)

* $A \times B =$
  $\{(1,2), (2,2), (3,2), (4,2), (1,5), (2,5), (3,5), (4.5)\}$
* $B \times A$:
  $\{(2.1), (2.2), (2.3), (2,4), (5,1), (5,2), (5,3), (5,4)\}$
* $A \cup (B \times C) =$
  $\{1,2,3,4, (2,3), (2,4), (2,7), (5,3), (5,4), (5,7)\}$
* $(A \cup B) \times C = (A \times C) \cup (B \times C) =$
  $\{(1,3), (2,3), (3,3), (4,3), (5,3),$
  $(1,4), (2,4), (3,4), (4,4), (5,4),$
  $(1,7), (2,7), (3,7), (4,7), (5,7)\}$

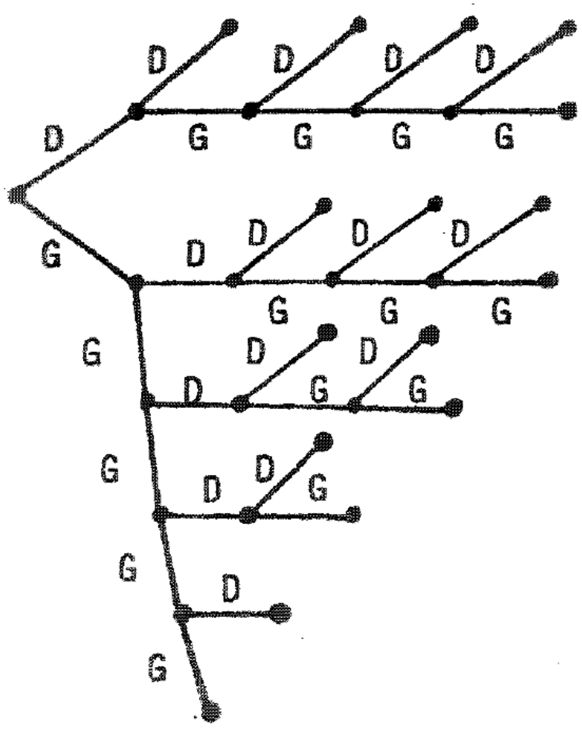
Ex 5.1: (3)

a) $|A \times B| = |A||B| = 9$
b) 2^9
c) 2^9
d) 2^7
e) $\binom{9}{5}$
f) $\binom{9}{7} + \binom{9}{8} + \binom{9}{9}$

Ex 5.1: (6)



Ex 5.1: (8)



Ex 5.1: (12)

* $2^{3|B|} = 4096 \Longrightarrow 3|B| = 12 \Longrightarrow |B| = 4.$

Ex 5.2: (4)

* $3^{|A|} = 2187 \Longrightarrow |A| = 7.$

Ex 5.2: (8)

a) True
b) False: Let $a = 1.5$. Then $\lfloor 1.5 \rfloor = 1 \neq 2 = \lceil 1.5 \rceil$
c) True
d) False: Let $a = 1.5$. Then $-\lceil a \rceil = -2 \neq -1 = \lceil -a \rceil$

Ex 5.2: (15)

a) One-to-one. The range is the set of all odd integers.
b) One-to-one. Range = Q.
c) Since $f(1) = f(0), f$ is not one-to-one. The range of $f = \{0, \pm 6, \pm 24, \pm 60, \ldots\} = \{n^3 - n | n \in \mathbb{Z}\}$.
d) One-to-one. Range = $(0, +\infty) = \mathbb{R}^+$.
e) One-to-one. Range = $[-1,1]$.
f) Since $f(\frac{\pi}{4}) = f(\frac{3\pi}{4}), f$ is not one-to-one. The range of $f = [0,1]$.

Ex 5.2: (20)

* The number of injective (or, one-to-one) functions from A to B is $(|B|!)/(|B|-5)! = 6720$, and $|B| = 8$.

Ex 5.2: (27.a)

$$A(1,3) = A(0, A(1,2)) = A(1,2) + 1 = A(0, A(1,1)) + 1 = [A(1,1) + 1] + 1$$
$$= A(1,1) + 2 = A(0, A(1,0)) + 2 = [A(1,0) + 1] + 2 = A(1,0) + 3$$
$$= A(0,1) + 3 = (1 + 1) + 3 = 5$$
$$A(2,3) = A(1, A(2,2))$$
$$A(2,2) = A(1, A(2,1))$$
$$A(2,1) = A(1, A(2,0)) = A(1, A(1,1))$$
$$A(1,1) = A(0, A(1,0)) = A(1,0) + 1 = A(0,1) + 1 = (1 + 1) + 1 = 3$$
$$A(2,1) = A(1,3) = A(0, A(1,2)) = A(1,2) + 1 = A(0, A(1,1))$$
$$= [A(1,1) + 1] + 1 = 5$$
$$A(2,2) = A(1,5) = A(0, A(1,4)) = A(1,4) + 1 = A(0, A(1,3)) + 1$$
$$= A(1,3) + 2 = A(0, A(1,2)) + 2 = A(1,2) + 3 = A(0, A(1,1)) + 3$$
$$= A(1,1) + 4 = 7$$
$$A(2,3) = A(1,7) = A(0, A(1,6)) = A(1,6) + 1 = A(0, A(1,5)) + 1$$
$$= A(0,7) + 1 = (7 + 1) + 1 = 9$$

Ex 5.2: (27.b)

Since $A(1,0) = A(0,1) = 2 = 0 + 2$, the result holds for the case where $n = 0$. Assuming the truth of the (open) statement for some k (≥ 0) we have $A(1, k) = k + 2$. Then we find that $A(1, k + 1) = A(0, A(1, k)) = A(1, k) + 1 = (k + 2) + 1 = (k + 1) + 2$, so the truth at $n = k$ implies the truth at $n = k + 1$. Consequentiy, $A(1, n) = n + 2$ for all $n \in \mathrm{N}$ by the Principle of Mathematical Induction.

Ex 5.2: (27.c)

Here we find that $A(2,0) = A(1,1) = 1 + 2 = 3$ (by the result in part(b)). So $A(2,0) = 3 + 2 \cdot 0$ and the given (open) statement is true in this first case. Next we assume the result true for some $k(\geq 0)$ - that is, we assume that $A(2,k) = 3 + 2k$. For $k + 1$ we then find that $A(2,k+1) = A(1,A(2,k)) = A(2,k) + 2$ (by part (b))$=$ $(3 + 2k) + 2$ (by the induction hypothesis)$= 3 + 2(k + 1)$. Consequently, for all $n \in \mathrm{N}, A(2,n) = 3 + 2n$ - by the Principle of Mathematical Induction.

Ex 5.2: (27.d)

Once again we consider what happens for n=0. Since $A(3,0) = A(2,1) = 3 + 2(1)$ (by part (c)) $= 5 = 2^{0+3} - 3$, the result holds in this first case. So now we assume the given (open) statement is true for some k (≥ 0) and this gives us the induction hypothesis: $A(3,k) = 2^{k+3} - 3$. For $n = k + 1$ it then follows that $A(3,k+1) = A(2,A(3,k)) = 3 + 2A(3,k)$ (by part (c))$= 3 + 2(2^{k+3} - 3)$ (by the induction hypothesis)$= 2^{(k+1)+3} - 3$, so the result holds for $n = k + 1$ whenever it does for $n = k$. Therefore, $A(3,n) = 2\hat{}(n + 3) - 3$, for all $n \in$ N - by the Principle of Mathematical Induction.

Ex 5.3: (1)

* $Let\ A = \{1,2,3,4\}, B = \{v, w, x, y, z\}$:

a) $f = \{(1, v), (2, v), (3, w), (4, x)\}$

b) $f = \{(1, v), (2, x), (3, y), (4, z)\}$

c) $Let\ A = \{1,2,3,4,5\}, B = \{w, x, y, z\}$,
$f = \{(1, w), (2, w), (3, x), (4, y), (5, z)\}$.

d) $Let\ A = \{1,2,3,4\}, B = \{w, x, y, z\}$,
$f = \{(1, w), (2, x), (3, y), (4, z)\}$

Ex 5.3: (4)

a) $6^4; \frac{6!}{2!}; 0$

b) $4^6; (4!)S(6,4); 0$

Ex 5.3: (8)

* Let A be the set of compounds and B the set of assistants. Then the number of assignments with no idle assistants is the number of onto functions from set A to set B. There are $5!\,S(9,5)$ such functions.

Ex 5.3: (12)

a) Since $31{,}100{,}905 = 5 \times 11 \times 17 \times 29 \times 31 \times 37$, we find that there are $S(6,3) = 90$ unordered factorizations of 31,100,905 into three factors - each greater than 1.

b) If the order of the factors in part (a) is considered relevant then there are $(3!)S(6,3) = 540$ such factorizations.

c) $\sum_{i=2}^{6} S(6,i) = S(6,2) + S(6,3) + S(6,4) + S(6,5) + S(6,6) = 202$

d) $\sum_{i=2}^{6} (i!)S(6,i) = (2!)S(6,2) + (3!)S(6,3) + (4!)S(6,4) + (5!)S(6,5) + (6!)S(6,6) = 4682$

Ex 5.3: (16)

a) (i) 10!
(ii) The given outcome - namely, $\{C_2, C_3, C_7\}, \{C_1, C_4, C_9, C_{10}\}, \{C_5\}, \{C_6, C_8\}$ - is an example of a distribution of ten distinct objects among four distinct containers, with no container left empty. [Or it is an example of an onto function $f: A \rightarrow B$ where $A = \{C_1, C_2, \dots, C_{10}\}$ and $B = \{1,2,3,4\}$.] There are $4!\,S(10,4)$ such distributions [or functions].
The answer to the question is $\sum_{i=1}^{10} i!\,S(10,i)$.
(iii) $\binom{10}{3}\sum_{i=1}^{7} i!\,S(7,i)$.

b) $\binom{9}{2}\sum_{i=1}^{7} i!\,S(7,i)$.

c) For 0 ≤ k ≤ 9, the number of outcomes where C_3 is tied for first place with k other candidates is $\binom{9}{k}\sum_{i=1}^{9-k} i!\,S(9-k,i)$. [Part (b) above is the special case where $k = 3 - 1 = 2$.] Summing over the possible values of k we have the answer $\sum_{k=0}^{9}\binom{9}{k}\sum_{i=1}^{9-k} i!\,S(9-k,i)$

Ex 5.4: (1)

* Here we find, for example, that $f(f(a,b),c) = f(a,c) = c$, while $f(a,f(b,c)) = f(a,b) = a$, so f is not associative.

Ex 5.4: (2)

a) For all $a, b \in \text{R}, f(a,b) = \lceil a+b \rceil = \lceil b+a \rceil = f(b,a)$, because the real numbers are commutative under addition. Hence f is a commutative (closed) binary operation.

b) This binary operation is not associative. For example,
$f(f(3.2, 4.7), 6.4) = f(\lceil 3.2 + 4.7 \rceil, 6.4) = f(\lceil 7.9 \rceil, 6.4)$
$= f(8, 6.4) = \lceil 8 + 6.4 \rceil = \lceil 14.4 \rceil = 15$, while,
$f\left(3.2, f(4.7,6.4)\right) = f(3.2, \lceil 4.7 + 6.4 \rceil) = f(3.2, \lceil 11.1 \rceil)$
$= f(3.2, 12) = \lceil 3.2 + 12 \rceil = \lceil 15.2 \rceil = 16$.

c) There is no identity element. If $a \in \text{R} - Z$ then for any $b \in \text{R}$, $\lceil a + b \rceil \in \text{Z}$. So if x were the identity element we would have $a = f(a, x) = \lceil a + x \rceil$ with $a \in \text{R} - Z$ and $\lceil a + x \rceil \in \text{Z}$

Ex 5.4: (5)

a) 25

b) 5^{25}

c) 5^{25}

d) 5^{10}

Ex 5.4: (8)

- Each element in A is of the form 2^i for some $1 \leq \text{i} \leq 5$, and $\gcd(2^i, 2^5) = 2^i = \gcd(2^5, 2^i)$, so $2^5 = 32$ is the identity element for f.

Ex 5.4: (12)

a) $\pi_A(D) = [0, +\infty);\ \pi_B(D) = R$
b) $\pi_A(D) = R;\ \pi_B(D) = [-1,1]$
c) $\pi_A(D) = [-1,1];\ \pi_B(D) = [-1,1]$

Ex 5.5: (2)

* The result follows by the Pigeonhole Principle where the eight people are the pigeons and the pigeonholcs are the seven days of the week.

Ex 5.5: (6)

- Any selection of size 101 from S must contain two consecutive integers $n, n+1$ and $\gcd(n, n+1) = 1$.

Ex 5.5: (13)

- Consider the subsets A of S where $1 \leq |A| \leq 3$. Since $|S| = 5$, there are $\binom{5}{1} + \binom{5}{2} + \binom{5}{3} = 25$ such subsets A. Let s_A denote the sum of the elements in A. Then $1 \leq s_A \leq 7 + 8 + 9 = 24$. So by the Pigeonhole Principle, there are two subsets of S whose elements yield the same sum.

Ex 5.5: (14)

* For $1 \leq i \leq 42$, let x_i count the total number of resumés Brace has sent out from the start of his senior year to the end of the i-th day. Then $1 \leq x_1 < x_2 < \cdots < x_{42} \leq 60$, and $x_1 + 23 < x_2 + 23 < \ldots < x_{42} + 23 \leq 83$. We have 42 distinct numbers $x_1, x_2, \ldots, x_{42}$, and 42 other distinct numbers $x_1 + 23, x_2 + 23, \ldots, x_{42} + 23$, all between 1 and 83 inclusive. By the Pigeonhole Principle $x_i = x_j + 23$ for some $1 \leq j < i \leq 42$; $x_i - x_j = 23$.

Ex 5.6: (7)

a) $(f \circ g)(x) = 3x - 1; (g \circ f)(x) = 3(x-1);$

$(g \circ h)(x) = \begin{cases} 0, x\ even \\ 3, x\ odd \end{cases}; (h \circ g)(x) = \begin{cases} 0, x\ even \\ 1, x\ odd \end{cases}$

$\big(f \circ (g \circ h)\big)(x) = f\big((g \circ h)(x)\big) = \begin{cases} -1, x\ even \\ 2, x\ odd \end{cases}$

$\big((f \circ g) \circ h\big)(x) = \begin{cases} (f \circ g)(0), x\ even \\ (f \circ g)(1), x\ odd \end{cases} = \begin{cases} -1, x\ even \\ 2, x\ odd \end{cases}$

b) $f^2(x) = f\big(f(x)\big) = x - 2; f^3(x) = x - 3;$

$g^2(x) = 9x; g^3(x) = 27x; h^2 = h^3 = h^{500} = h$

Ex 5.6: (10)

a) $f^{-1} = \{(x, y)|2y + 3x = 7\}$
b) $f^{-1} = \{(x, y)|ay + bx = c, b \neq 0, a \neq 0\}$
c) $f^{-1} = \left\{(x, y)\middle|y = x^{\frac{1}{3}}\right\} = \{(x, y)|x = y^3\}$
d) Here $f(0) = f(-1) = 0$, so f is not one-to-one, and consequently f is not invertible.

Ex 5.6: (16)

a) $[0,2)$
b) $[-1,2)$
c) $[0,1)$
d) $[0,2)$
e) $[-1,3)$
f) $[-1,0) \cup [2,4)$

Ex 5.6: (17.a~17.g)

a) The range of $f = \{2,3,4,\dots\} = Z^+ - \{1\}$.
b) Since 1 is not in the range of f. The function is not onto.
c) For all $x, y \in \mathrm{Z}^+, f(x) = f(y) \Rightarrow x + 1 = y + 1 \Rightarrow x = y$, so f is one-to-one.
d) The range of g is Z^+.
e) Since $g(Z^+) = Z^+$, the codomain of g, this function is onto.
f) Here $g(1) = 1 = g(2)$, and $1 \neq 2$, so g is not one-to-one.
g) For all $x \in \mathrm{Z}^+, (g \circ \mathrm{f})(x) = g\big(f(x)\big) = g(x + 1) = \max\{1, (x + 1) - 1\} = \max\{1, x\} = x, since\ x \in \mathrm{Z}^+$. Hence $g \circ \mathrm{f} = 1_{Z^+}$.

Ex 5.6: (17.h & 17.i)

h) $(f \circ g)(2) = f(\max\{1,1\}) = f(1) = 1 + 1 = 2$
$(f \circ g)(3) = f(\max\{1,2\}) = f(2) = 2 + 1 = 3$
$(f \circ g)(4) = f(\max\{1,3\}) = f(3) = 3 + 1 = 4$
$(f \circ g)(7) = f(\max\{1,6\}) = f(6) = 6 + 1 = 7$
$(f \circ g)(12) = f(\max\{1,11\}) = f(11) = 11 + 1 = 12$
$(f \circ g)(25) = f(\max\{1,24\}) = f(24) = 24 + 1 = 25$

i) No, because the functions f, g are not inverses of each other. The calculations in part (h) may suggest that $f \circ g = 1_{Z^+}$ since $(f \circ g)(x) = x \; for \; x \geq 2$. But we also find that $(f \circ g)(1) = f(\max\{1,0\}) = f(1) = 2$, so $(f \circ g)(1) \neq 1$, and, consequently, $f \circ g \neq 1_{Z^+}$.

Ex 5.6: (22)

- It follows from Theorem 5.11 that there are $5!$ Invertible functions $f: A \to B$.