

SageMath 1: Using SageMath



Cheng-Hsin Hsu

National Tsing Hua University

Department of Computer Science

What is SageMath

- Sage stands for *System for Algebra and Geometry Experimentation*
- Starts from a **Computer Algebra System (CAS)**
- Now a full spectrum of math tools, including
 - **Algebra**
 - Combinatorics
 - Matrix Computations
 - Numerical Math
 - Calculus
 - **Number Theory**
 - Visualization
 - Many other topics



<http://www.sagemath.org>

How is SageMath Built

- Created in 2005 by William Stein at University of Washington
- Goal: Create an affordable math software to solve various math problems
- Approach
 - Leverage **opensource** (GPL-ed) software, such as *SciPy, Maxima, and GAP*
 - Adopt the popular **Python** as the programming language
- **Outcome: A huge set of math tools!**

What SageMath Can Do

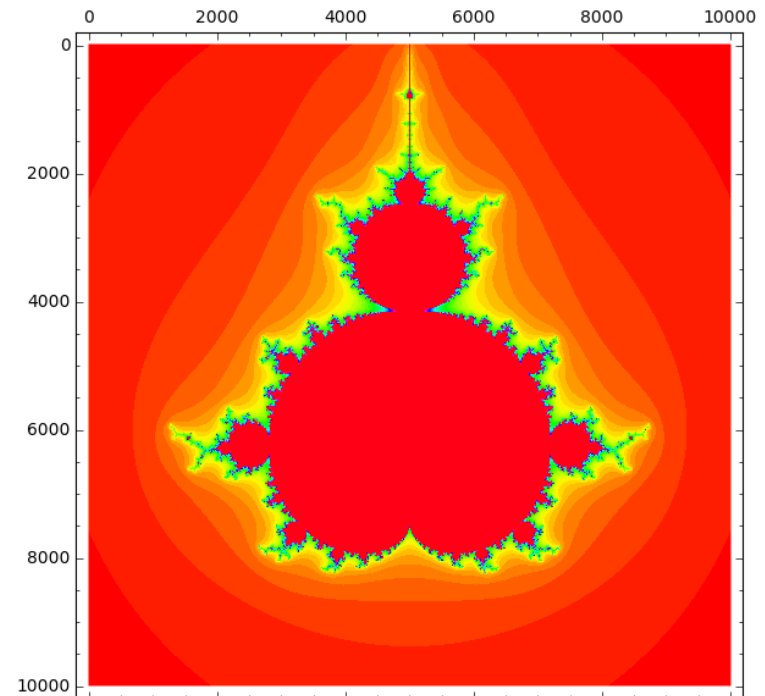
- Solving exact problems

```
⊕ ⓘ  
integral(x*sin(x^2),x)  
-1/2*cos(x^2)  
⊕ ⓘ  
integral(x*sin(x^2),x, -1, 1)  
0
```

- Solving numerical problems

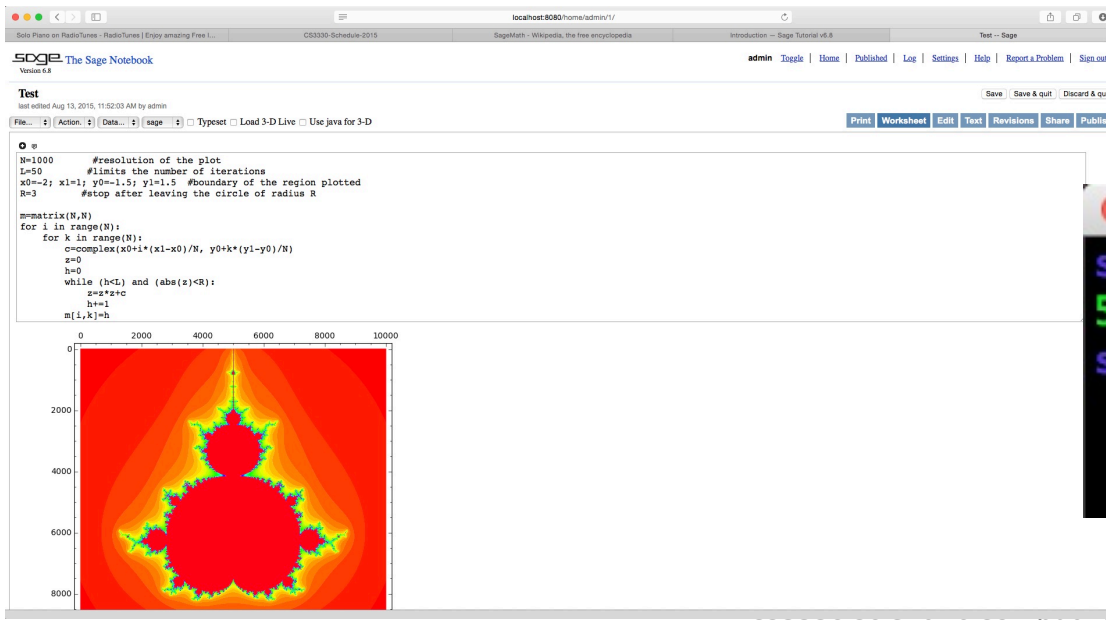
```
⊕ ⓘ  
numerical_integral(x*sin(x^2), -1, 1)  
(0.0, 5.103670946680643e-15)
```

- Plot cool figures
 - Get a sense of more global behavior



Four Ways to Use SageMath

- **Notebook**: Web-based interface
- **Interactive command-line**: Python-like shell prompt
- **Programs**: Write and compile Sage programs
- **Scripts**: Invoke Sage libraries from Python

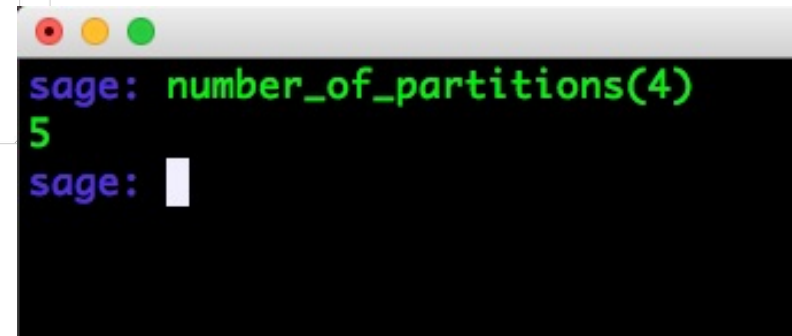


The screenshot shows the Sage Notebook web interface. The top navigation bar includes links for 'admin', 'Toggle', 'Home', 'Published', 'Log', 'Settings', 'Help', 'Report a Problem', and 'Sign out'. Below the navigation bar, there are buttons for 'Print', 'Worksheet', 'Edit', 'Text', 'Revisions', 'Share', and 'Publish'. The main content area displays a code cell with the following Python code:

```
0 #
N=1000 #resolution of the plot
L=50 #limits the number of iterations
x0=-2; x1=1; y0=-1.5; y1=1.5 #boundary of the region plotted
R=3 #stop after leaving the circle of radius R

M=matrix(N,N)
for l in range(N):
    for k in range(N):
        c=complex(x0+l*(x1-x0)/N, y0+k*(y1-y0)/N)
        z=0
        h=0
        while (h<L) and (abs(z)<R):
            z=z**c
            h+=1
        m[l,k]=h
```

Below the code cell, a fractal plot is displayed. The plot shows a complex, self-similar fractal structure with a color gradient from red to yellow. The x-axis ranges from 0 to 10000, and the y-axis ranges from 0 to 8000.



```
sage: number_of_partitions(4)
5
sage: |
```

Where Are Notebook and Command-line

start here

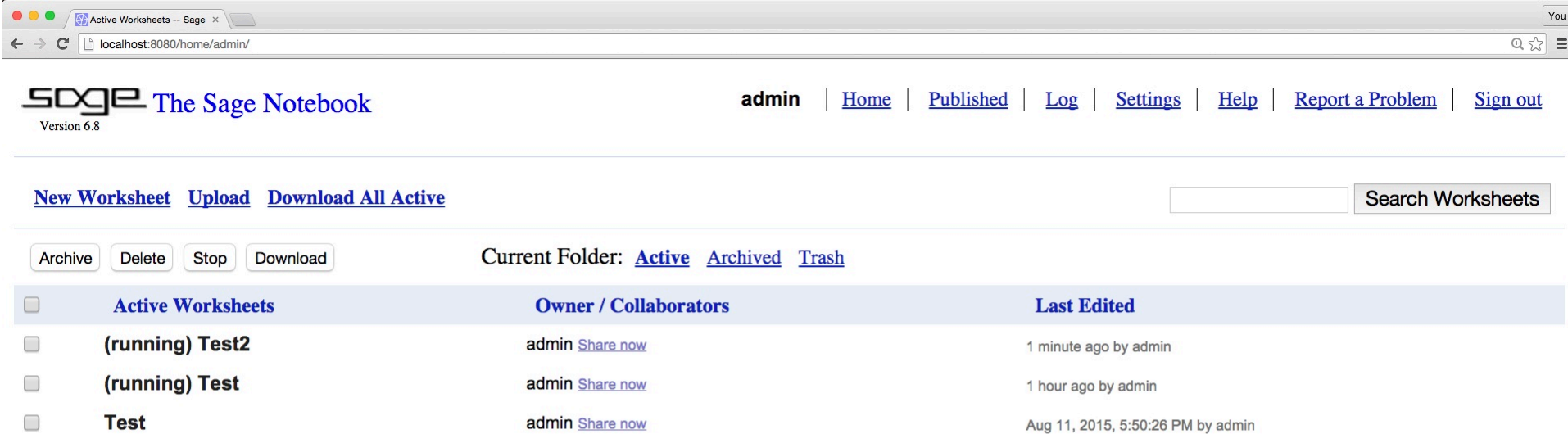
The screenshot shows a Beamer presentation titled "SageMath 1: Using SageMath" by Cheng-Hsin Hsu from National Tsing Hua University. The presentation content includes:

- Slide 1:** Title slide with author and affiliation.
- Slide 2:** "What is SageMath" - Sage stands for System for Algebra and Geometry Experimentation. Starts from a Computer Algebra System (CAS). New a full spectrum of math tools, including:
 - Algebra
 - Combinatorics
 - Matrix Computations
 - Numerical Math
 - Calculus
 - Number Theory
 - Visualization
 - Many other topics
- Slide 3:** "How is SageMath Built" - Created in 2005 by William Stein at University of Washington. Goal: Create an affordable math software to solve various math problems. Approach: Leverage open-source (GPL-ed) software, such as Solly, Maxima, and GUP. Adapt the popular Python as the programming language. Outcome: A huge set of math tools!
- Slide 4:** "What SageMath Can Do" - Solving exact problems, Solving numerical problems, Plot cool figures.
- Slide 5:** "How Do We Use SageMath"

The presentation is displayed in a window titled "Sage1_Using_SageMath.pptx". A red box highlights the "Terminal Session" option in the "Sage" menu. Another red box highlights the Sage icon in the dock.

start here

SageMath Notebook



The screenshot shows the SageMath Notebook web interface. At the top, there is a navigation bar with the Sage logo and the text "The Sage Notebook Version 6.8". To the right of the logo are several links: "admin", "Home", "Published", "Log", "Settings", "Help", "Report a Problem", and "Sign out". Below the navigation bar, there are three main links: "New Worksheet", "Upload", and "Download All Active". To the right of these links is a search box labeled "Search Worksheets". Below the search box, there are four buttons: "Archive", "Delete", "Stop", and "Download". To the right of these buttons, it says "Current Folder: [Active](#) [Archived](#) [Trash](#)". Below this, there is a table with three columns: "Active Worksheets", "Owner / Collaborators", and "Last Edited".

Active Worksheets	Owner / Collaborators	Last Edited
<input type="checkbox"/> (running) Test2	admin Share now	1 minute ago by admin
<input type="checkbox"/> (running) Test	admin Share now	1 hour ago by admin
<input type="checkbox"/> Test	admin Share now	Aug 11, 2015, 5:50:26 PM by admin

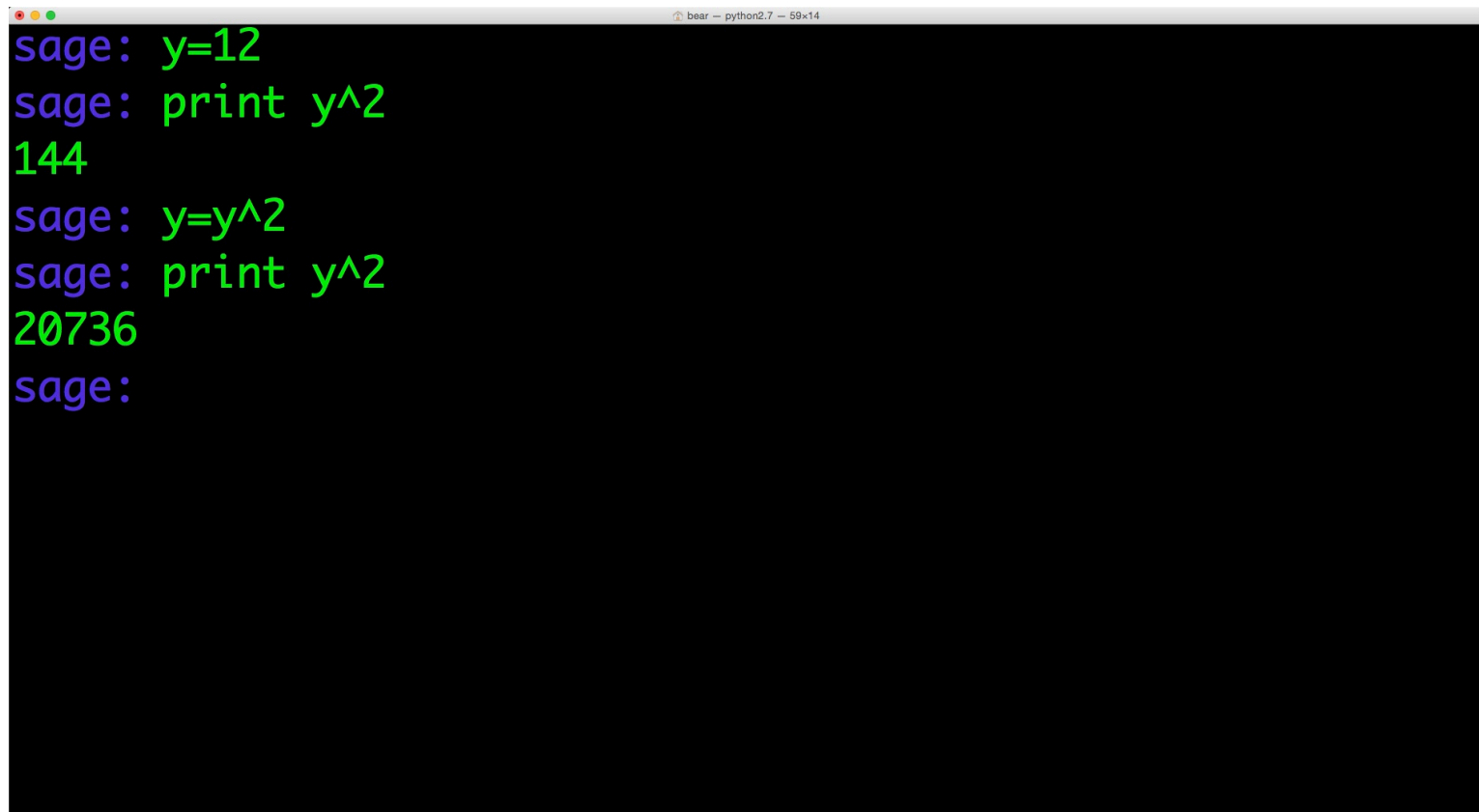
- Each notebook has several **worksheets**
- Worksheets can be saved, opened, downloaded, shared, and etc.
- SageCloud (<https://cloud.sagemath.com>) is the next generation of Web-based interface ← We won't cover it

SageMath Worksheet

The screenshot shows the SageMath Notebook interface. At the top, there's a navigation bar with the Sage logo and "The Sage Notebook" text. Below that, there are several links: admin, Toggle, Home, Published, Log, Settings, Help, Report a Problem, and Sign out. The main content area is titled "Test" and shows a worksheet with two cells. The first cell contains the command `n(sqrt(123))` and its output `11.0905365064094`. The second cell contains the command `arccos(1/3)` and is highlighted with a blue border. Below the second cell is an "evaluate" button. The interface also includes a toolbar with buttons for File, Action, Data, sage, Typeset, Load 3-D Live, Use java for 3-D, Print, Worksheet, Edit, Text, Revisions, Share, and Publish.

- Each worksheet contains several (independent, stateless) **cells**
- Each cell is a box where you can type in formulas and **evaluate** them ← by pressing the **evaluate** button or **Shift-Enter**
- You can always come back and edit your formulas/commands and **re-evaluate** them

SageMath Commandline



```
sage: y=12
sage: print y^2
144
sage: y=y^2
sage: print y^2
20736
sage:
```

- Command-line remembers states

Use SageMath as a Calculator



Introduction

Save Save & quit Discard & quit

last edited Aug 13, 2015, 3:28:17 PM by admin

File... Action... Data... sage Typeset Load 3-D Live Use java for 3-D

Print Worksheet Edit Text Revisions Share Publish



2+3

5



$900 * (1 + 0.05 * (90 / 365))$

911.095890410959



$10000 * (1 + 0.12)^3$

14049.2800000000



evaluate



Square Root Function

Introduction Save Save & quit Discard & quit

last edited Aug 13, 2015, 3:28:17 PM by admin

File... Action... Data... sage Typeset Load 3-D Live Use java for 3-D

Print Worksheet Edit Text Revisions Share Publish

+ $\sqrt{25}$
5

+ $\sqrt{8}$
 $2\sqrt{2}$ ← SageMath loves exact answers

+ $N(\sqrt{8})$
2.82842712474619 ← Numerical approximation

+ $64^{1/6}$
2

Special Constants

Introduction (Sage) x You
localhost:8080/home/admin/3/

Introduction

last edited Aug 13, 2015, 3:28:17 PM by admin

Save Save & quit Discard & quit

File... Action... Data... sage Typeset Load 3-D Live Use java for 3-D

Print Worksheet Edit Text Revisions Share Publish

+ ⓘ
pi*2
2*pi ← Simplest exact value

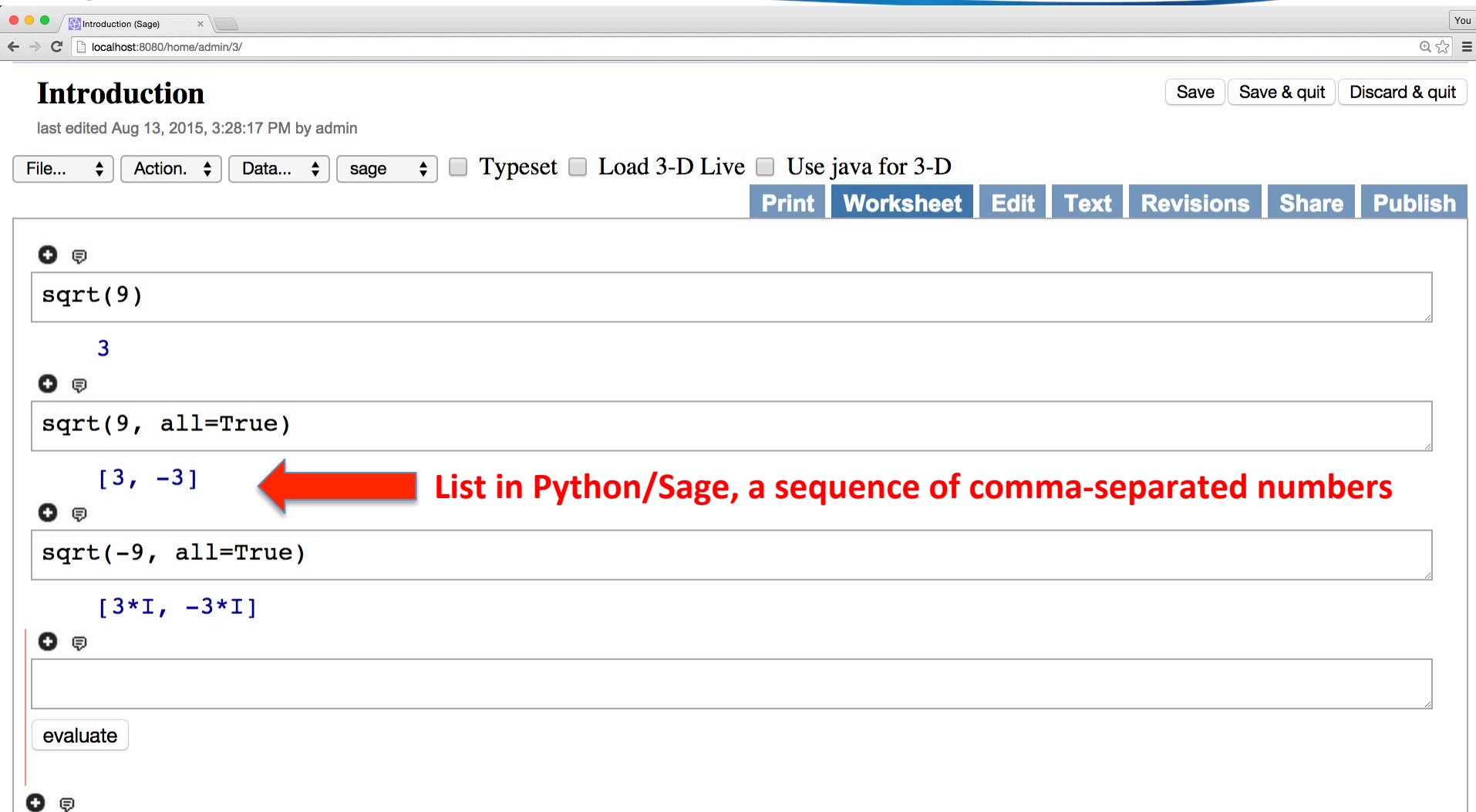
+ ⓘ
e^2
e^2

+ ⓘ
e^2.5
12.1824939607035 ← Decimals automatically lead to numerical approximation

+ ⓘ
N(pi, prec=256)
3.1415926535897932384626433832795028841971693993751058209749445923078164\
06286

+ ⓘ

More on Square Roots



The screenshot shows a SageMath web interface. At the top, there's a browser window with the URL `localhost:8080/home/admin/3/`. Below the browser, the page title is "Introduction" and it says "last edited Aug 13, 2015, 3:28:17 PM by admin". There are buttons for "Save", "Save & quit", and "Discard & quit". Below that, there are dropdown menus for "File...", "Action...", "Data...", and "sage", along with checkboxes for "Typeset", "Load 3-D Live", and "Use java for 3-D". A navigation bar contains buttons for "Print", "Worksheet", "Edit", "Text", "Revisions", "Share", and "Publish".

The main content area shows three code input boxes with their corresponding outputs:

- Code: `sqrt(9)`
Output: `3`
- Code: `sqrt(9, all=True)`
Output: `[3, -3]`
- Code: `sqrt(-9, all=True)`
Output: `[3*I, -3*I]`

A red arrow points from the text "List in Python/Sage, a sequence of comma-separated numbers" to the output `[3, -3]`.

At the bottom left, there is an "evaluate" button and a small icon.

SageMath is Case Sensitive

The screenshot shows a SageMath web interface with a browser window titled "Introduction (Sage)". The address bar shows "localhost:8080/home/admin/3/". The page title is "Introduction" and it was last edited on Aug 13, 2015, at 3:28:17 PM by admin. The interface includes a toolbar with buttons for "File...", "Action.", "Data...", "sage", "Typeset", "Load 3-D Live", and "Use java for 3-D". There are also buttons for "Print", "Worksheet", "Edit", "Text", "Revisions", "Share", and "Publish".

The main content area displays four code blocks with their outputs:

```
sQrt(9)
```

Traceback (click to the left of this block for traceback)
...
NameError: name 'sQrt' is not defined

Mostly, with four exceptions: *true, false, i, n*

```
true
```

True

```
3*i - 3*I
```

0

```
n(sqrt(17)) - N(sqrt(13))
```

0.517554350153671

How to Get Help

Introduction

last edited Aug 13, 2015, 3:28:17 PM by admin

File... Action... Data... sage Typeset Load 3-D Live Use java for 3-D

Print Worksheet Edit

+ ⌘

log?

File: /Applications/Sage-6.8.app/Contents/Resources/sage/local/lib/python2.

Type: <class 'sage.functions.log.Function_log'>

Definition: log(*args, **kwds)

Docstring:

The natural logarithm of x. See *log?* for more information about its beh

```
Last login: Thu Aug 13 16:35:13 on ttys0
Bears-iMac:~ bear$ /Applications/Sage-6.
| SageMath Version 6.8, Release Date: 20
| Type "notebook()" for the browser-base
| Type "help()" for help.
sage: help(log)
sage: 
```

- Notebook: *command?*
- Command-line: *help(command)*
- Also, utilize the *Tab-completions*

Trigonometry

Introduction (Sage) x
localhost:8080/home/admin/3/

Introduction Save Save & quit Discard & quit

last edited Aug 13, 2015, 3:28:17 PM by admin

File... Action... Data... sage Typeset Load 3-D Live Use java for 3-D

Print Worksheet Edit Text Revisions Share Publish



`sin(pi/3)`

`1/2*sqrt(3)`



`N(cos(pi/12))`

`0.965925826289068`



`arccos(1/2)`

`1/3*pi`



`cos(pi/6)-sin(pi/3)`

`0`

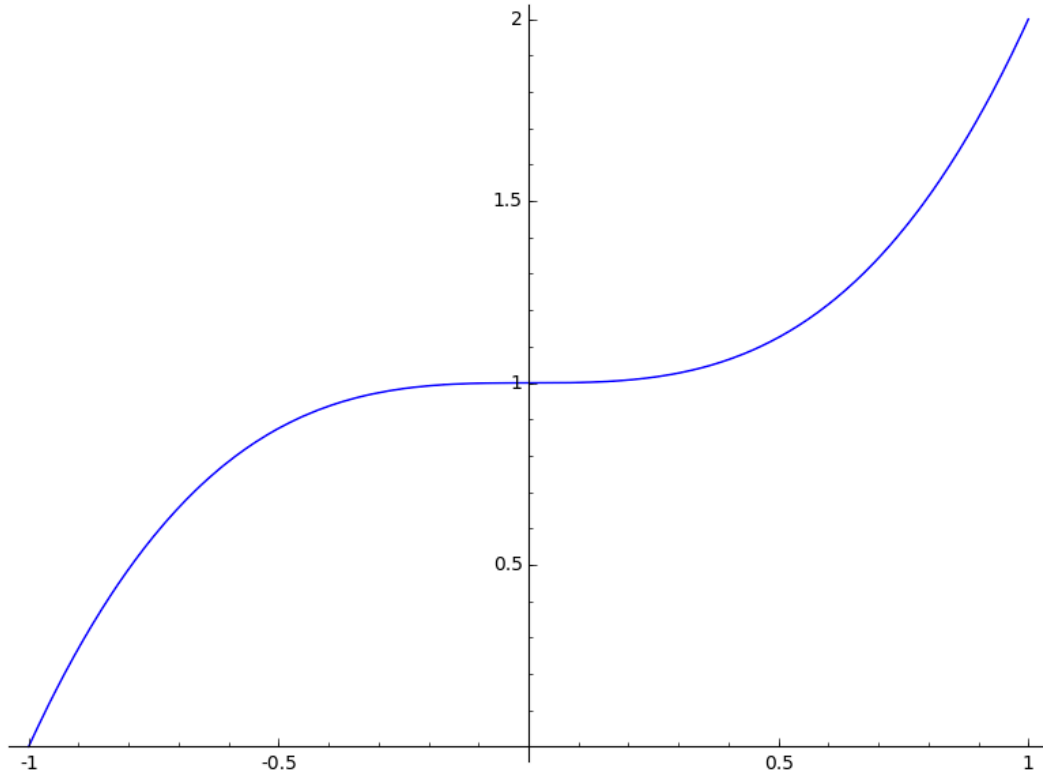


Simple 2D Plots



```
plot(x^3+1)
```

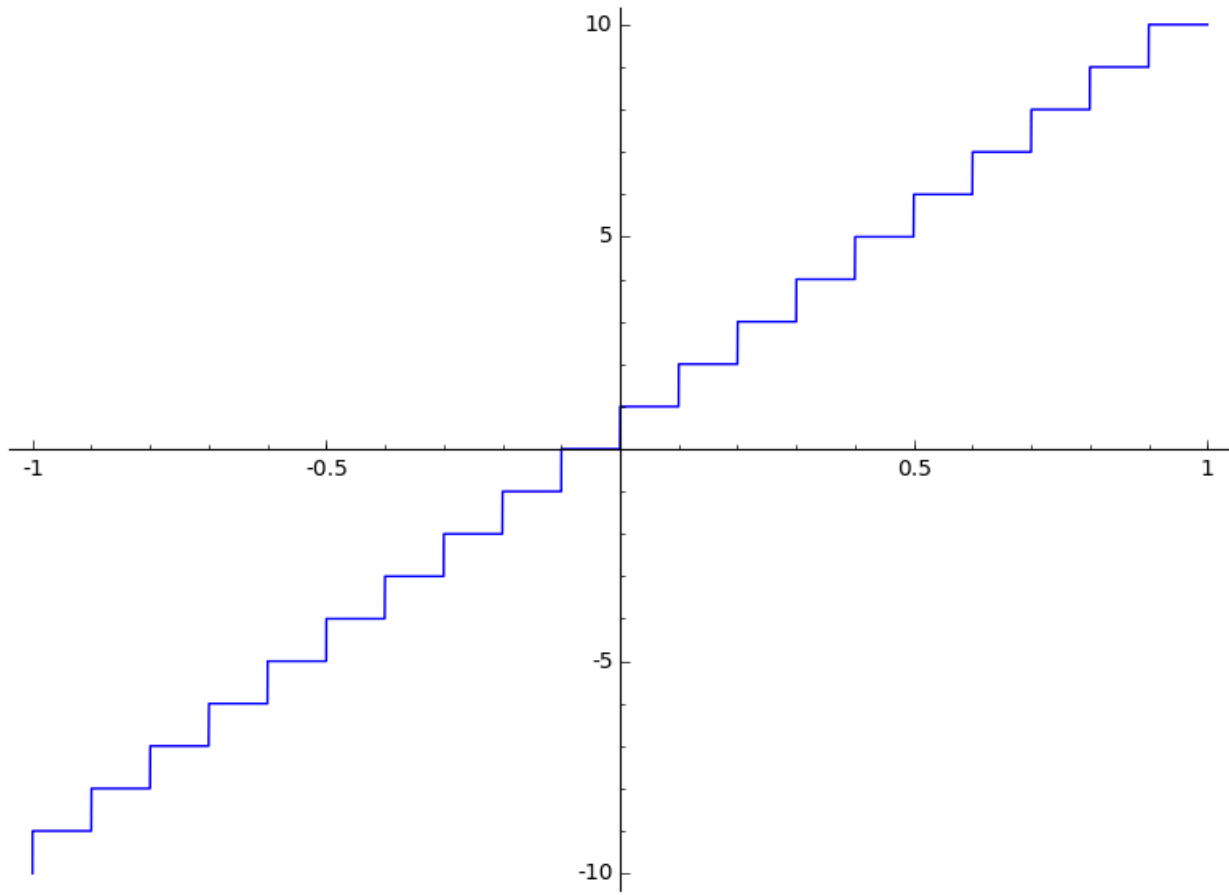
evaluate



More Complicated Function



```
plot(ceil(x*10))
```

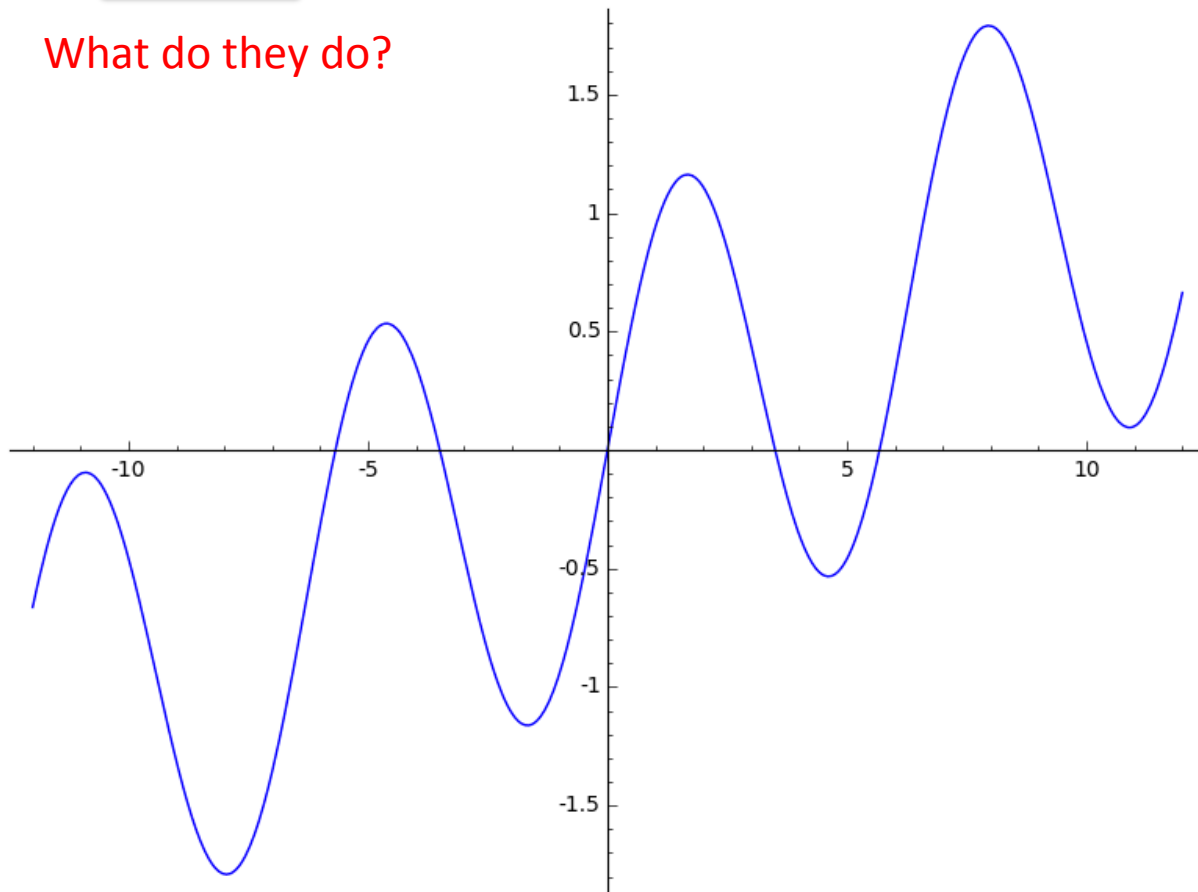


Even More Complicated



```
plot(sin(x)+0.1*x, -12, 12)
```

What do they do?

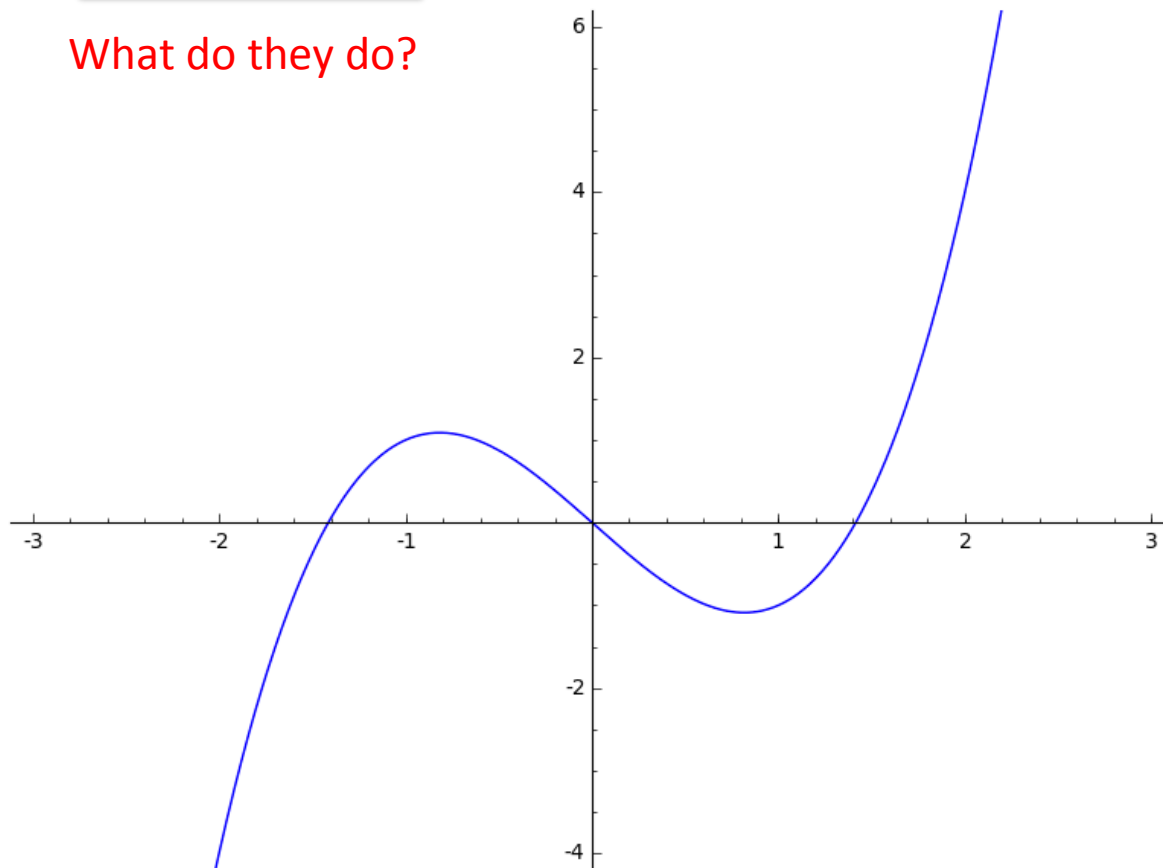


Ranges of Axes



```
plot(x^3-2*x, -3, 3, ymin=-4, ymax=6)
```

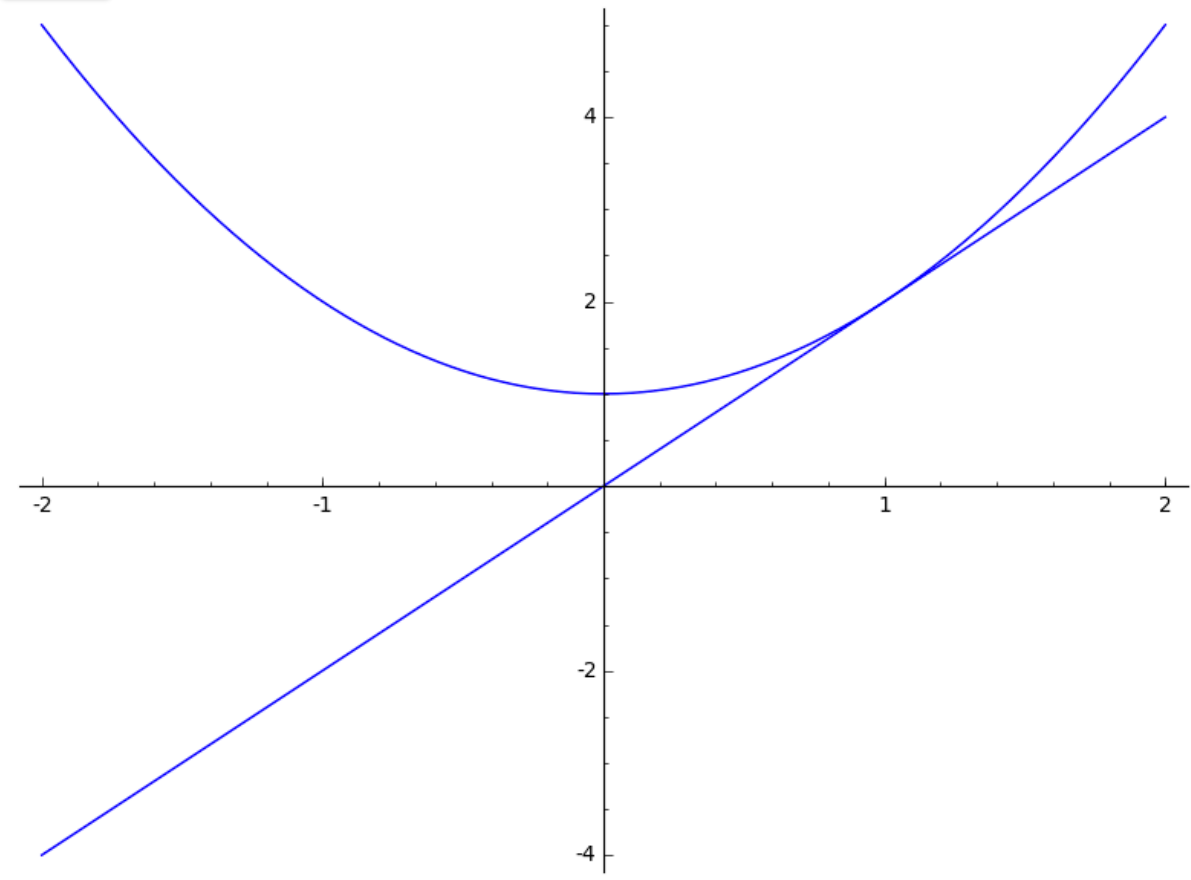
What do they do?



Superimposing of Multiple Graph



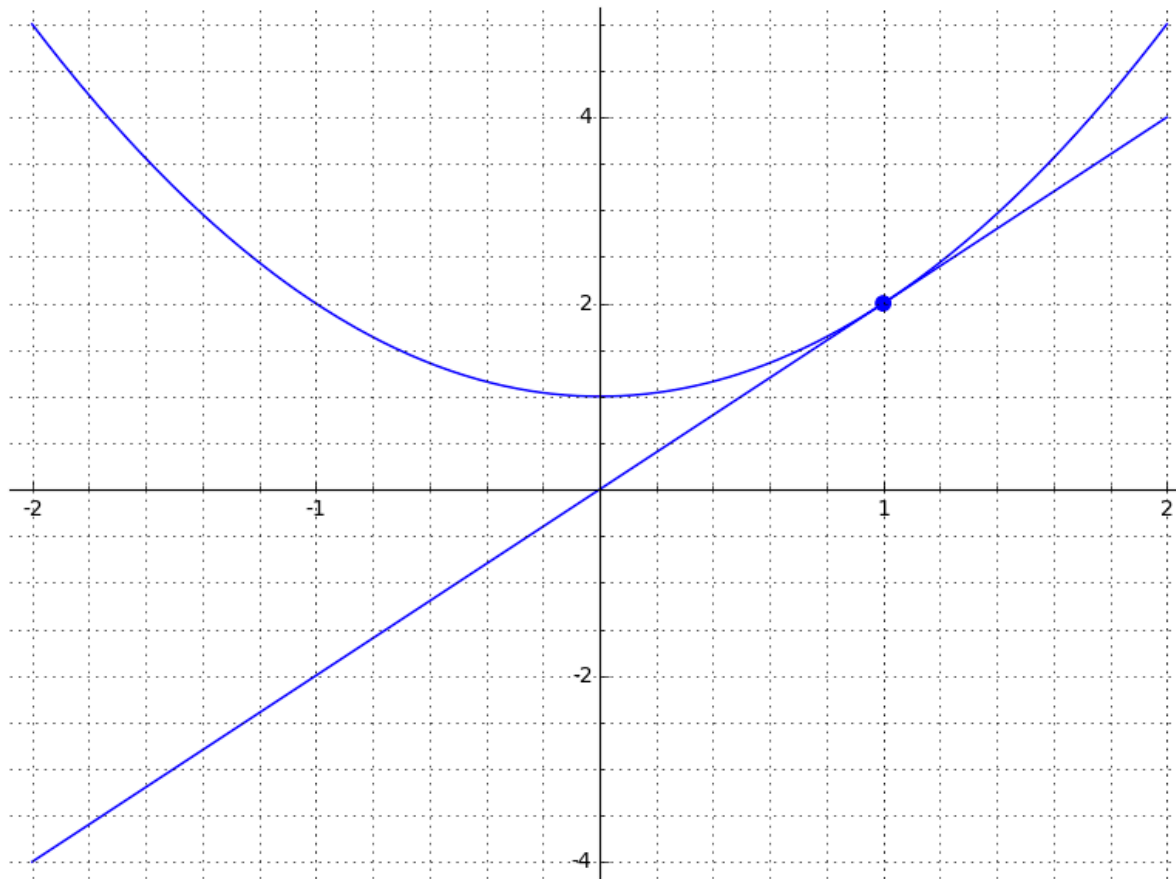
```
plot(2*x, -2, 2) + plot(x^2+1, -2,2)
```



Polished Graph



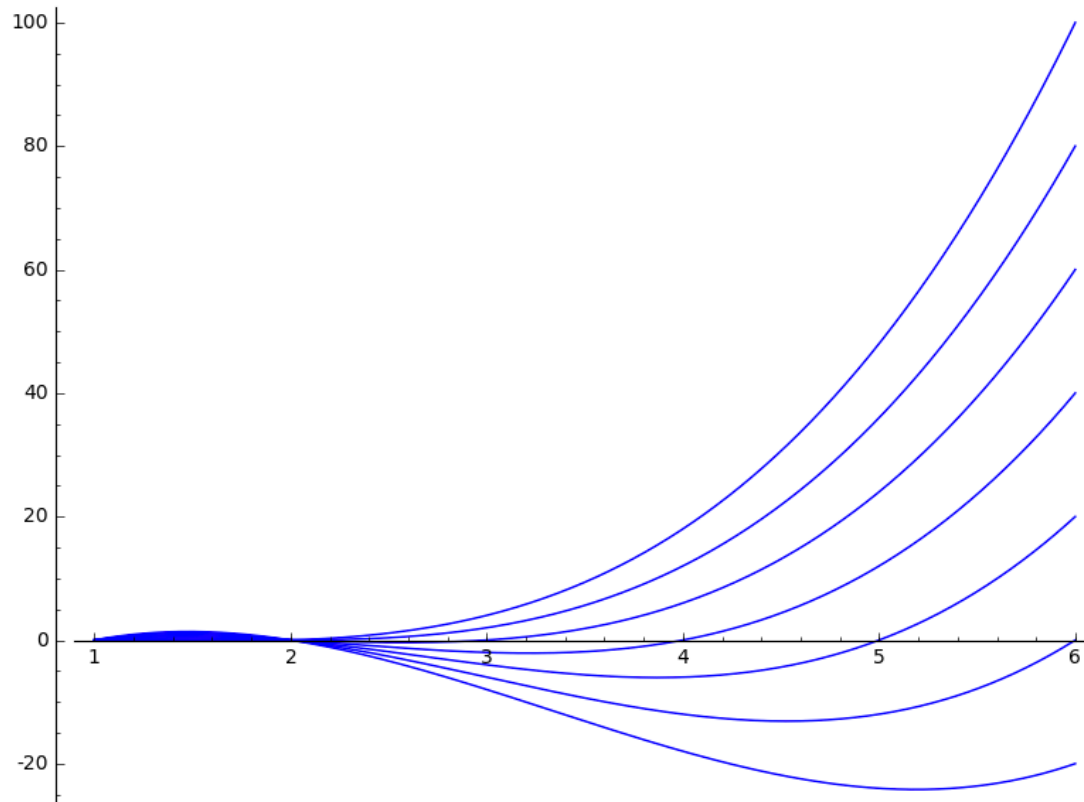
```
plot(2*x, -2, 2, gridlines='minor') + plot (x^2+1, -2,2) + point ((1,2), size=60)
```



Last Example on Graphs



```
plot((x-1)*(x-2)*(x-7), 1,6)+plot((x-1)*(x-2)*(x-6), 1,6)+plot((x-1)*(x-2)*(x-5),  
1,6)+plot((x-1)*(x-2)*(x-4), 1,6)+plot((x-1)*(x-2)*(x-3), 1,6)+plot((x-1)*(x-2)*(x-2),  
1,6)+plot((x-1)*(x-2)*(x-1), 1,6)
```



Reduced Row Echelon Form (RREF)

- System of equations:

$$\begin{aligned} 3x - 4y + 5z &= 14 \\ x + y - 8z &= -5 \\ 2x + y + z &= 7 \end{aligned} \quad \longrightarrow \quad A = \left[\begin{array}{ccc|c} 3 & -4 & 5 & 14 \\ 1 & 1 & -8 & -5 \\ 2 & 1 & 1 & 7 \end{array} \right]$$

- RREF:

$$A' = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array} \right] \longrightarrow x = 3, y = 0, z = 1$$

Getting RREF in SageMath

- Create a matrix ← what are the first two arguments of *matrix(.)*?
- Invoke *rref(.)* function



```
A=matrix(3, 4, [3, -4, 5, 14, 1, 1, -8, -5, 2, 1, 1, 7])  
print A.rref()
```

evaluate

```
[1 0 0 3]  
[0 1 0 0]  
[0 0 1 1]
```

Practice, a Tricky Question

- Try to solve the following equation system using SageMath

$$2x - 5z + y = 6 + w$$

$$5 + z - y = 0$$

$$w + 3(x + y) = z$$

$$1 + 2x - y = w - 3x$$

- Answer:

$$w = \frac{-107}{7}, x = \frac{-12}{7}, y = \frac{54}{7}, z = \frac{19}{7}$$

Initializing Matrices

- Two ways to create matrices



```
B = matrix(2, 3, [1, 2, 3, 4, 5, 6])  
print B
```

```
[1 2 3]  
[4 5 6]
```



```
B = matrix([ [1, 2, 3], [4, 5, 6] ])  
print B
```

```
[1 2 3]  
[4 5 6]
```

Exceptional RREF

$$\begin{array}{l} x + 2y + 3z = 7 \\ 4x + 5y + 6z = 16 \\ 7x + 8y + 9z = 24 \end{array} \quad \longrightarrow \quad C1 = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 4 & 5 & 6 & 16 \\ 7 & 8 & 9 & 24 \end{array} \right]$$



```
B = matrix(3, 4, [1, 2, 3, 7, 4, 5, 6, 16, 7, 8, 9, 24])  
print B.rref()
```

```
[ 1  0 -1  0]  
[ 0  1  2  0]  
[ 0  0  0  1]
```

- What is the solution?

Exceptional RREF (cont.)

$$\begin{array}{l} x + 2y + 3z = 7 \\ 4x + 5y + 6z = 16 \\ 7x + 8y + 9z = 25 \end{array} \quad \longrightarrow \quad C2 = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 4 & 5 & 6 & 16 \\ 7 & 8 & 9 & 25 \end{array} \right]$$



```
B = matrix(3, 4, [1, 2, 3, 7, 4, 5, 6, 16, 7, 8, 9, 25])
print B.rref()
```

```
[ 1  0 -1 -1]
[ 0  1  2  4]
[ 0  0  0  0]
```

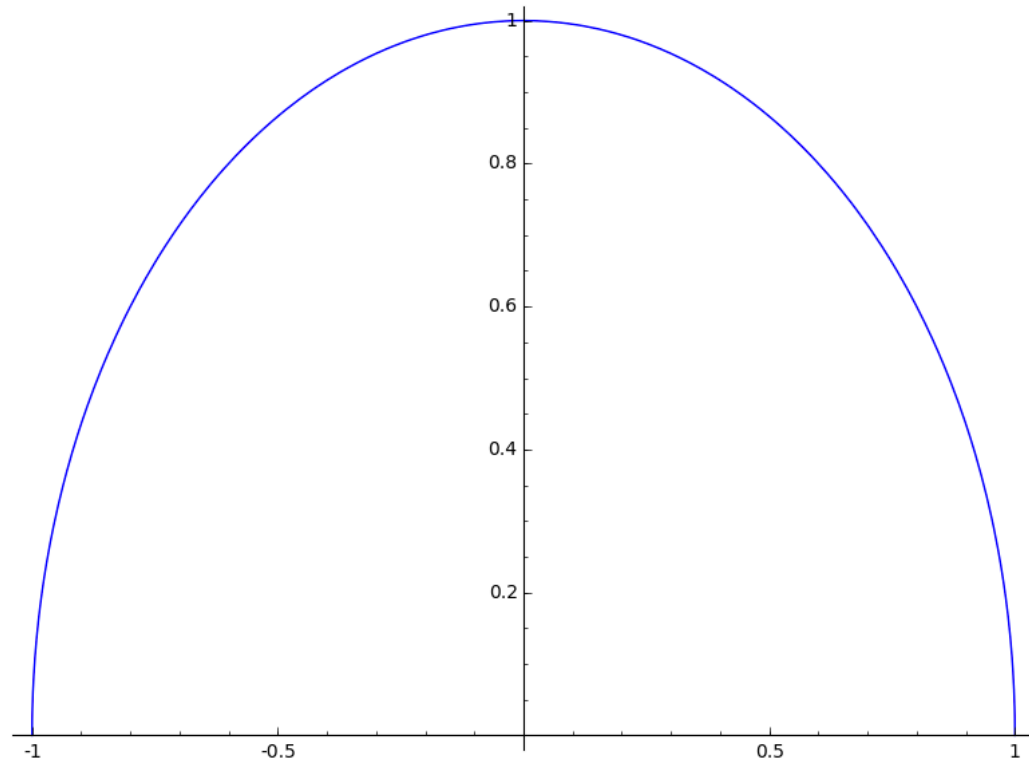
- What are the solutions?

Define our Own Functions



```
f(x)=sqrt(1-x^2)
print(f(0.1))
print(f(0.001))
print(f(0.00001))
print(f(0.0000001))
print(f(0.000000001))
plot(f, -1, 1)
```

```
0.994987437106620
0.9999999499999875
0.99999999999950000
0.99999999999999995
1.00000000000000000
```



Function Composition



```
f(x) = 3*x+5
g(x) = x^3 + 1
h(x) = f(g(x))
print h
h(-2)
```

```
x | --> 3*x^3 + 8
-16
```



```
f(x) = 3*x+5
g(x) = x^3 + 1
h(x) = f(g(x))
print h
h2(x) = g(f(x))
print h2
```

```
x | --> 3*x^3 + 8
x | --> (3*x + 5)^3 + 1
```

Factorization



```
print factor(12345)
print factor(987654321)
print factor(x^2 - 8*x + 15)
print gcd(x^2 - 5*x + 6, x^2 - 8*x + 15)
```

```
3 * 5 * 823
3^2 * 17^2 * 379721
(x - 3)*(x - 5)
x - 3
```


Expanding a Function



```
a(x) = x^2 - 5 *x + 6
b(x) = x^2 - 8 *x + 15
f(x) = a(b(x))
print f(x)
print f.expand()
print f.factor()
```

```
(x^2 - 8*x + 15)^2 - 5*x^2 + 40*x - 69
x |--> x^4 - 16*x^3 + 89*x^2 - 200*x + 156
(x^2 - 8*x + 13)*(x - 2)*(x - 6)
```

Use Sage to Solve Problem Symbolically

- Solving $\frac{x^2}{2} - x - 2 = 0$

- Symbolically, we get

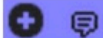
$$1 \pm \sqrt{5}$$

 We will start from here

- Numerically, we get

3.23606797749979 and -1.23606797749979

Single-Variable Formulas



```
solve(x^2 + 3*x + 2, x)
```

```
[x == -2, x == -1]
```



```
solve(x^2 + 2*x + 3, x)
```

```
[x == -I*sqrt(2) - 1, x == I*sqrt(2) - 1]
```



```
var('alpha')  
solve(cos(alpha) == 1/2, alpha)
```

```
[alpha == 1/3*pi]
```



Declare variable, notice that x is always a variable without explicit declaration

Multiple-Variable Formulas



```
var('a b c')  
solve(a*x^2 + b*x + c == 0, x)
```

```
[x == -1/2*(b + sqrt(b^2 - 4*a*c))/a, x == -1/2*(b - sqrt(b^2 -  
4*a*c))/a]
```



```
var('b')  
solve([x+b ==6, x-b == 4], x, b)
```

```
[[x == 5, b == 1]]
```



```
var('a, b, c')  
solve([9*a + 3*b + c == 32, 4*a + 2*b + c == 15, a + b + c ==6], a, b, c)
```

```
[[a == 4, b == -3, c == 5]]
```



What does it do? Remember how to solve the same problem using RREF?

Non-Linear Equation Systems



```
solve ((x-1) * (x-2) * (x-3) * (x-4) * (x-5) == 0, x)
```

```
[x == 1, x == 2, x == 3, x == 4, x == 5]
```



```
answers = solve ((x-1) * (x-2) * (x-3) * (x-4) * (x-5) == 0, x)
print answers[0]
print answers[3]
```

```
x == 1
x == 4
```



```
var('p q y')
eq1 = p+q == 9
eq2 = q*y + p*x == -6
eq3 = q*y^2 + p*x^2 == 24
eq4 = p == 2
solve([eq1, eq2, eq3, eq4], p, q, x, y)
```

List of list? How to make it easier to read?



evaluate

```
[[p == 2, q == 7, x == -1/3*sqrt(70) - 2/3, y ==
2/21*sqrt(7)*sqrt(5)*sqrt(2) - 2/3], [p == 2, q == 7, x == 1/3*sqrt(70)
- 2/3, y == -2/21*sqrt(7)*sqrt(5)*sqrt(2) - 2/3]]
```

Higher Order Equations



```
answers = solve([x^2 - y^2 == 1, (x^2)/4 + (y^2)/3 == 1], x, y)
print answers[0]
print answers[1]
print answers[2]
print answers[3]
```

```
[x == -4/7*sqrt(7), y == -3/7*sqrt(7)]
[x == -4/7*sqrt(7), y == 3/7*sqrt(7)]
[x == 4/7*sqrt(7), y == -3/7*sqrt(7)]
[x == 4/7*sqrt(7), y == 3/7*sqrt(7)]
```



```
solve(log( x^2 ) == 5/3, x)
```

evaluate

```
[x == -e^(5/6), x == e^(5/6)]
```



```
solve(sin(x+y)==0.5, x)
```

Try this to see what we get

```
[x == 1/6*pi - y]
```

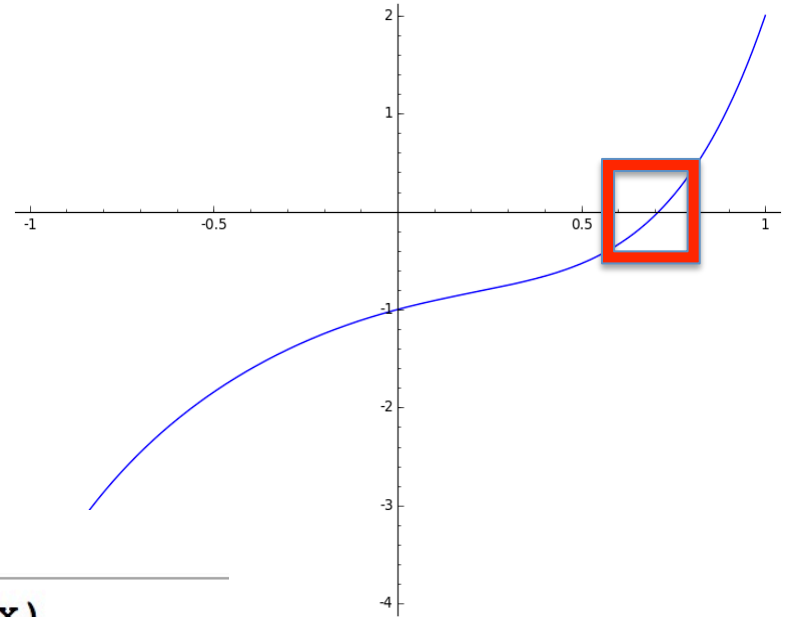


```
var('a0, a1, a2, a3, a4')
solve(a4*x^4 + a3*x^3 + a2*x^2 + a1*x + a0 == 0, x)
```



Quantic Polynomials and Numerical Solver

$$\text{Solve } x^5 + x^4 + x^3 - x^2 + x - 1 = 0$$



```
solve(x^5 + x^4 + x^3 - x^2 + x - 1 == 0, x)
```

```
[0 == x^5 + x^4 + x^3 - x^2 + x - 1]
```



Why can't we find a solution?



```
find_root(x^5 + x^4 + x^3 - x^2 + x - 1, -100, 100)
```

```
0.7104342557869131
```

More Numerical Solver Examples



```
find_root(e^x - 1/x, -1, 1)
```

0.5671432904097954



```
find_root(cos(x)==sin(x), 0, pi/2)
```

evaluate

0.7853981633974484



```
var('n')  
find_root(7000 == 5000*(1+0.045/12)^n, 0, 100000)
```

89.89406093308004



Exercise: what are we calculating?

How to Take Derivatives



```
diff(x^4 - x^2 + x , x)
```

$4x^3 - 2x + 1$



```
diff(sin(x/e), x)
```

$\cos(xe^{-1})e^{-1}$



```
g(x)=e^(-10*x)
g.derivative()
f(x) = g.derivative()
f(-12)
```

$-10e^{120}$

Higher Order Derivatives



```
diff(x^3 + 2*x^2 + 3, 2)
```

$6*x + 4$



```
derivative(x^3 + 2*x^2 + 3, 2)
```

$6*x + 4$



```
diff(x^3 + 2*x^2 + 3, 4)
```

0

Indefinite Integral



```
integral(x*cos(x^2), x)
```

$$\int x \cos(x^2) dx = \frac{1}{2} \sin(x^2) ?$$

```
1/2*sin(x^2)
```



```
integral(x / (x^2 + 2), x)
```

$$\int \frac{x}{x^2 + 2} dx = \frac{1}{2} \log(x^2 + 2) ?$$

```
1/2*log(x^2 + 2)
```

Don't forget the constant!

Definite and Impossible Integrals



```
integral(x*cos(x^2), x, -2, 3)
```

```
1/2*sin(9) - 1/2*sin(4)
```

$$\int_{-2}^3 x \cos(x^2) dx$$



```
integral(x / (x^2 + 2), x, 1, 2)
```

```
1/2*log(6) - 1/2*log(3)
```

$$\int_1^2 \frac{x}{x^2 + 2} dx$$



```
var('y')
```

```
assume(y>0)
```

```
integral(2 / sqrt(pi) * (e ^ (-x^2)), x, 0, y)
```

```
erf(y)
```



We even define a special function for this!

Numerical Integrations and Partial Fractions



```
integral(exp(-x^3) * sin(x^2), x, 1, 3)
```

```
integrate(e^(-x^3)*sin(x^2), x, 1, 3)
```



```
numerical_integral(x / (x^2 + 2), 1, 2)
```

```
(0.34657359027997264, 3.84773979655831e-15)
```



```
f(x) = (x^3 - x) / (x^2 + 5*x + 6)
```

```
f.partial_fraction()
```

```
x |--> x + 24/(x + 3) - 6/(x + 2) - 5
```

Summary

- We introduced SageMath, an opensource project based on Python
- We go over basics of SageMath, including symbolic and numerical solutions, matrices, and simple plots
- We will mostly use SageMath for **symbolic solutions**
 - SageMath was initially designed for this
 - Other tasks are done in Matlab/Octave
- References:
 - <http://www.sagemath.org> ← Official Web and resources
 - <http://www.gregorybard.com/SAGE.html> ← Our textbook

SageMath #1 Homework (S1)

1. (1%) Find at least 5 real number solutions of $e^{-x^2} \sin(8x) = 0$ in $[-3, 3]$ using SageMath
– Hint: Plot the curve first!
2. (1%) Run $A = \text{random_matrix}(\mathbb{Z}\mathbb{Z}, 3, 4)$ in SageMath. Map the matrix into an equation system (say variables are x , y , and z). Write down the equation system. Solve the equation system using SageMath.
3. (1%) Execute the following code in SageMath. Explain why the two prints give different answers? You need to identify the key difference to get the point.

```
x = ceil(random() * 999)
print sqrt(x)
print N(sqrt(x))
```