SageMath 1: Using SageMath

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What is SageMath

- Sage stands for *System for Algebra and Geometry Experimentation*
- Starts from a Computer Algebra System (CAS)
- Now a full spectrum of math tools, including
  - Algebra
  - Combinatorics
  - Matrix Computations
  - Numerical Math
  - Calculus
  - Number Theory
  - Visualization
  - Many other topics

http://www.sagemath.org

How is SageMath Built

- Created in 2005 by William Stein at University of Washington
- Goal: Create an affordable math software to solve various math problems
- Approach
  - Leverage opensource (GPL-ed) software, such as *SciPy, Maxima, and GAP*
  - Adopt the popular Python as the programming language
- Outcome: A huge set of math tools!

What SageMath Can Do

- Solving exact problems

```
integral(x*sin(x^2),x)
```

```
-1/2*cos(x^2)
```

```
integral(x*sin(x^2),x, -1, 1)
```

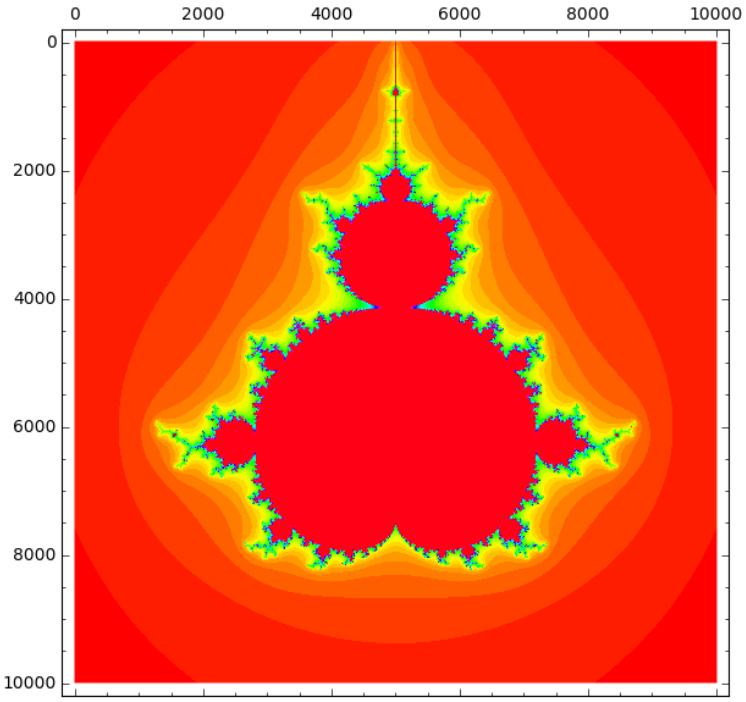
```
0
```

- Solving numerical problems

```
numerical_integral(x*sin(x^2), -1, 1)
```

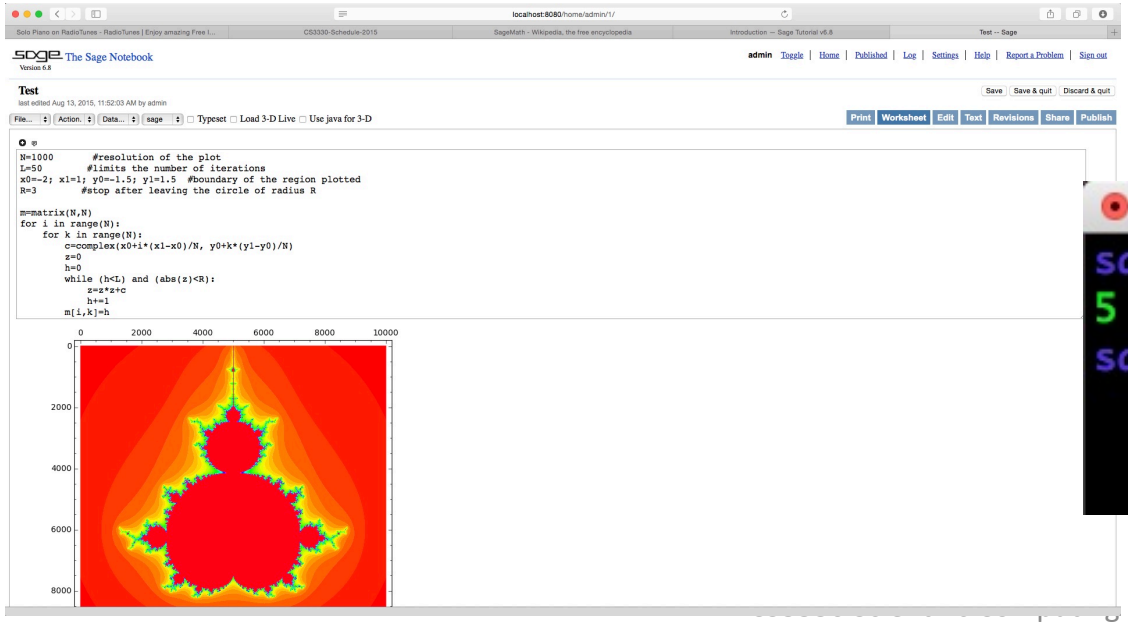
```
(0.0, 5.103670946680643e-15)
```

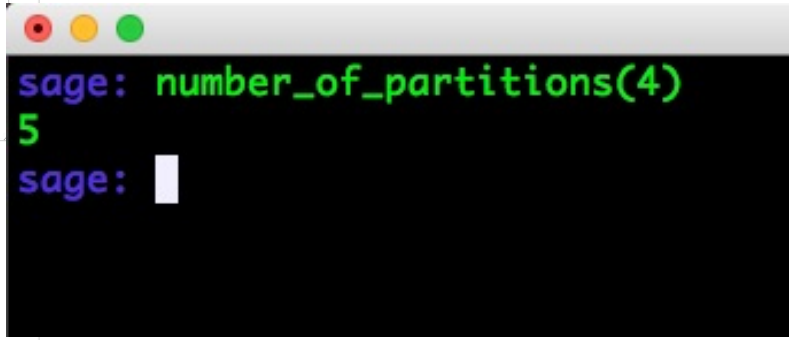
- Plot cool figures
  - Get a sense of more global behavior



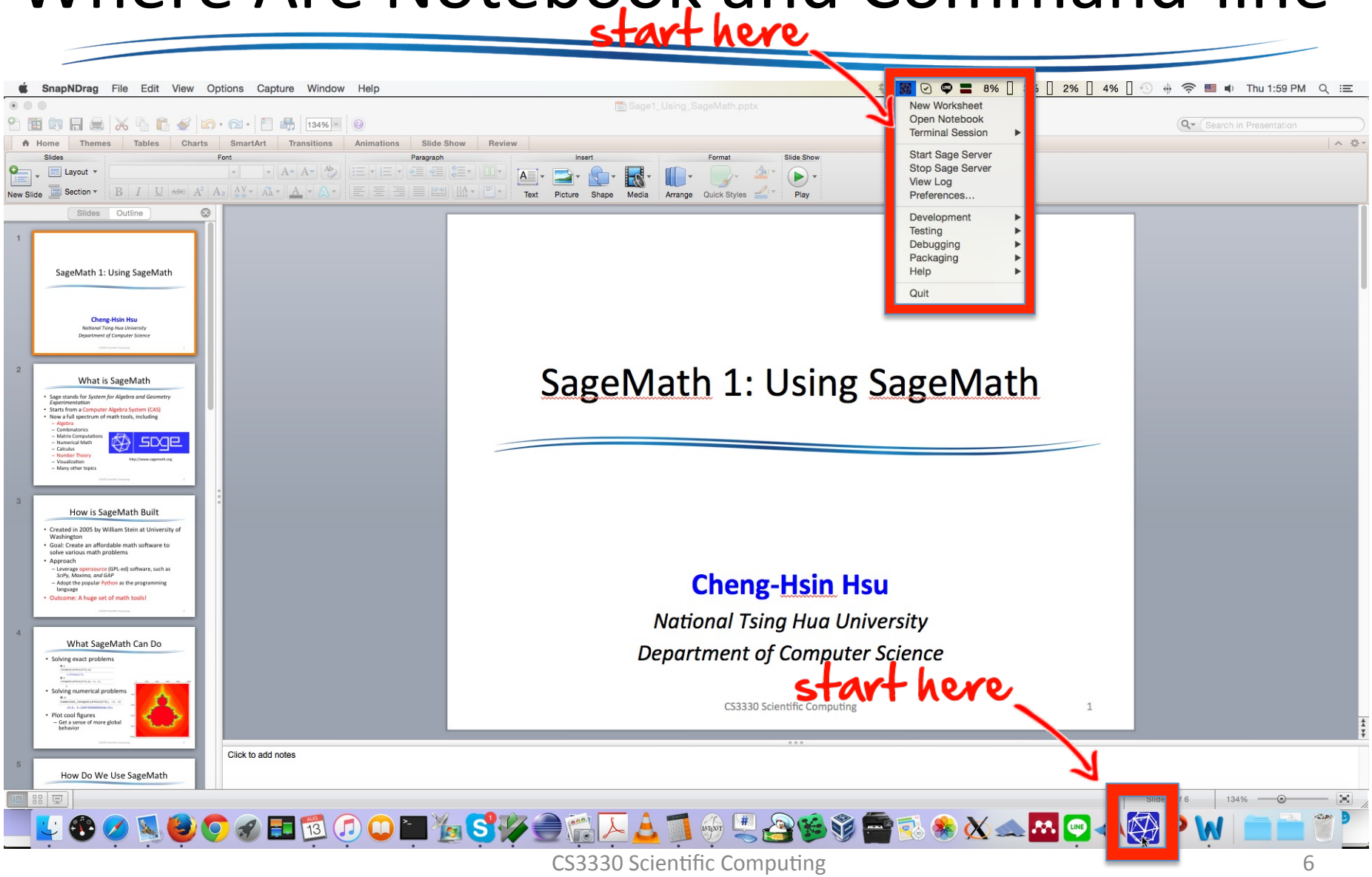
Four Ways to Use SageMath

- Notebook: Web-based interface
- Interactive command-line: Python-like shell prompt
- Programs: Write and compile Sage programs
- Scripts: Invoke Sage libraries from Python

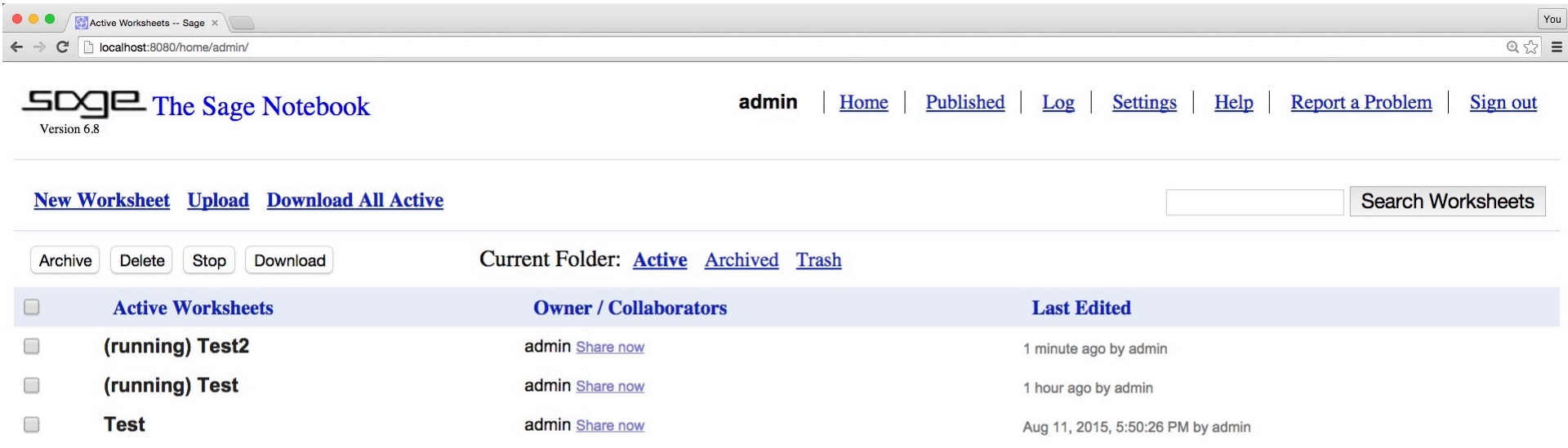




Where Are Notebook and Command-line

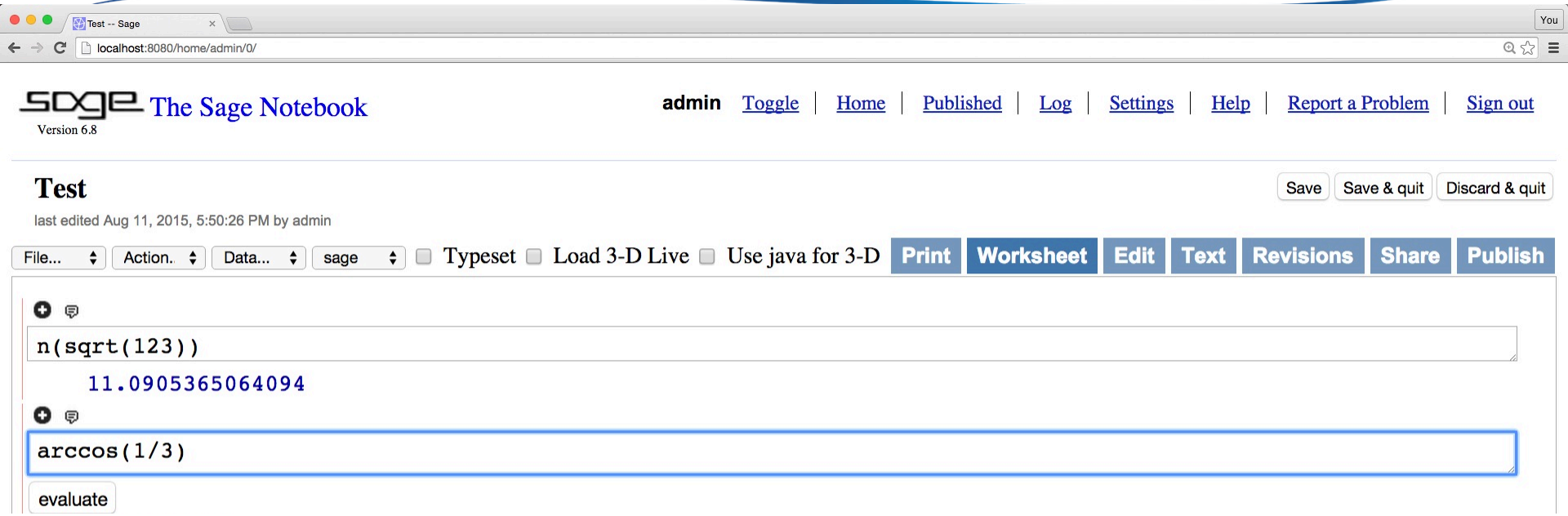


SageMath Notebook



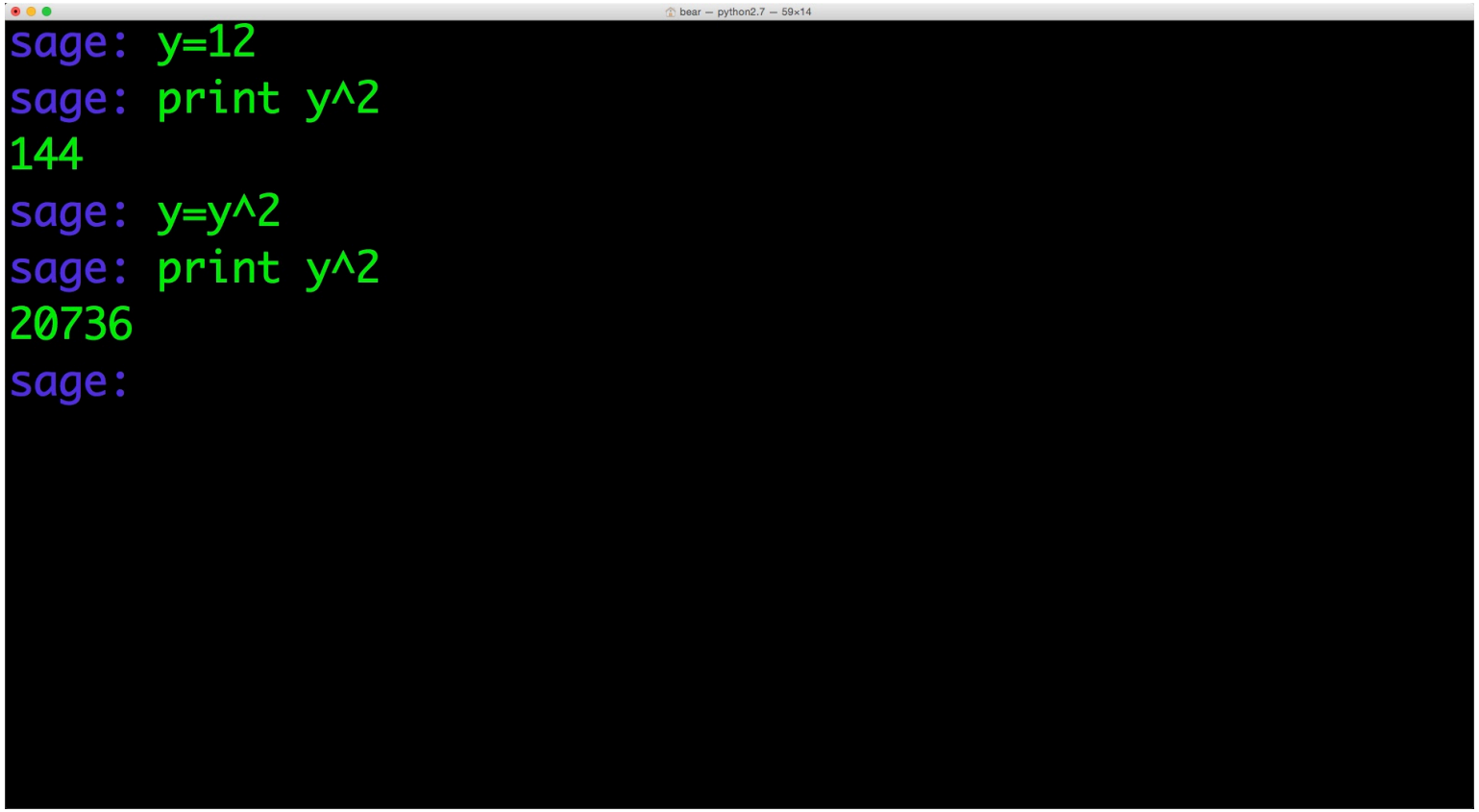
- Each notebook has several worksheets
- Worksheets can be saved, opens, downloaded, shared, and etc.
- SageCloud (https://cloud.sagemath.com) is the next generation of Web-based interface ← We won't cover it

SageMath Worksheet



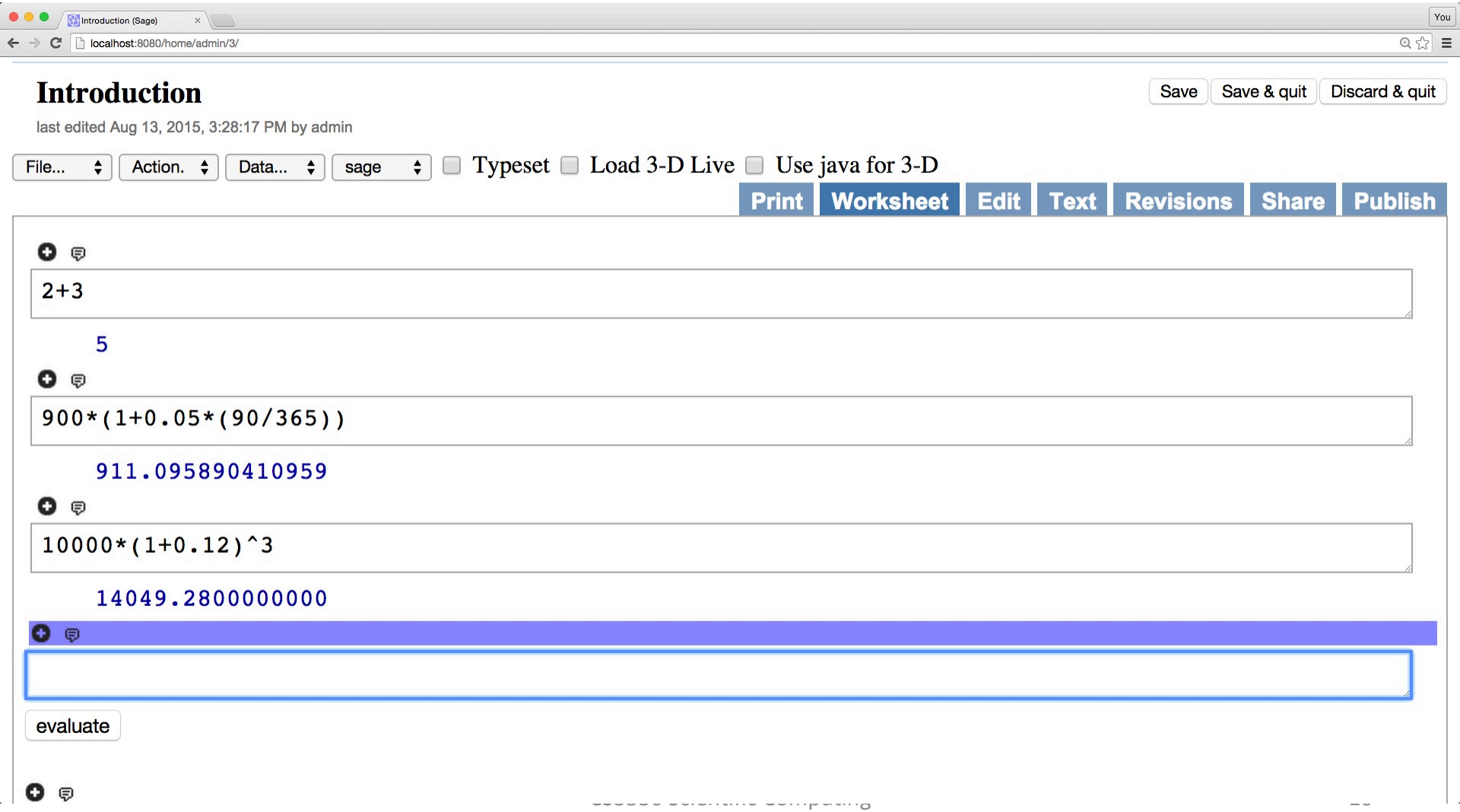
- Each worksheet contains several (independent, stateless) cells
- Each cell is a box where you can type in formulas and evaluate them ← by pressing the evaluate button or *Shift-Enter*
- You can always come back and edit your formulas/commands and re-evaluate them

SageMath Commandline

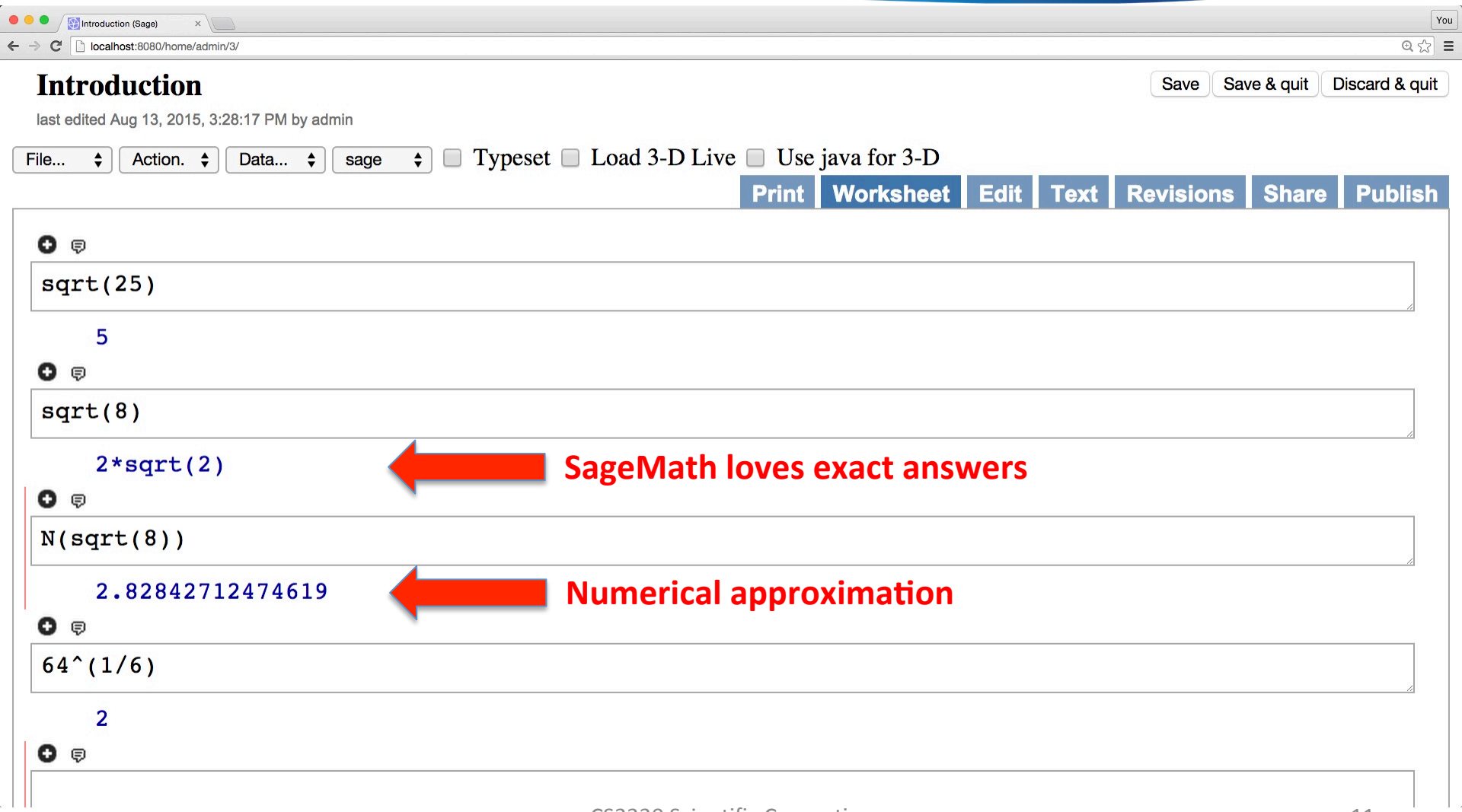


- Command-line remembers states

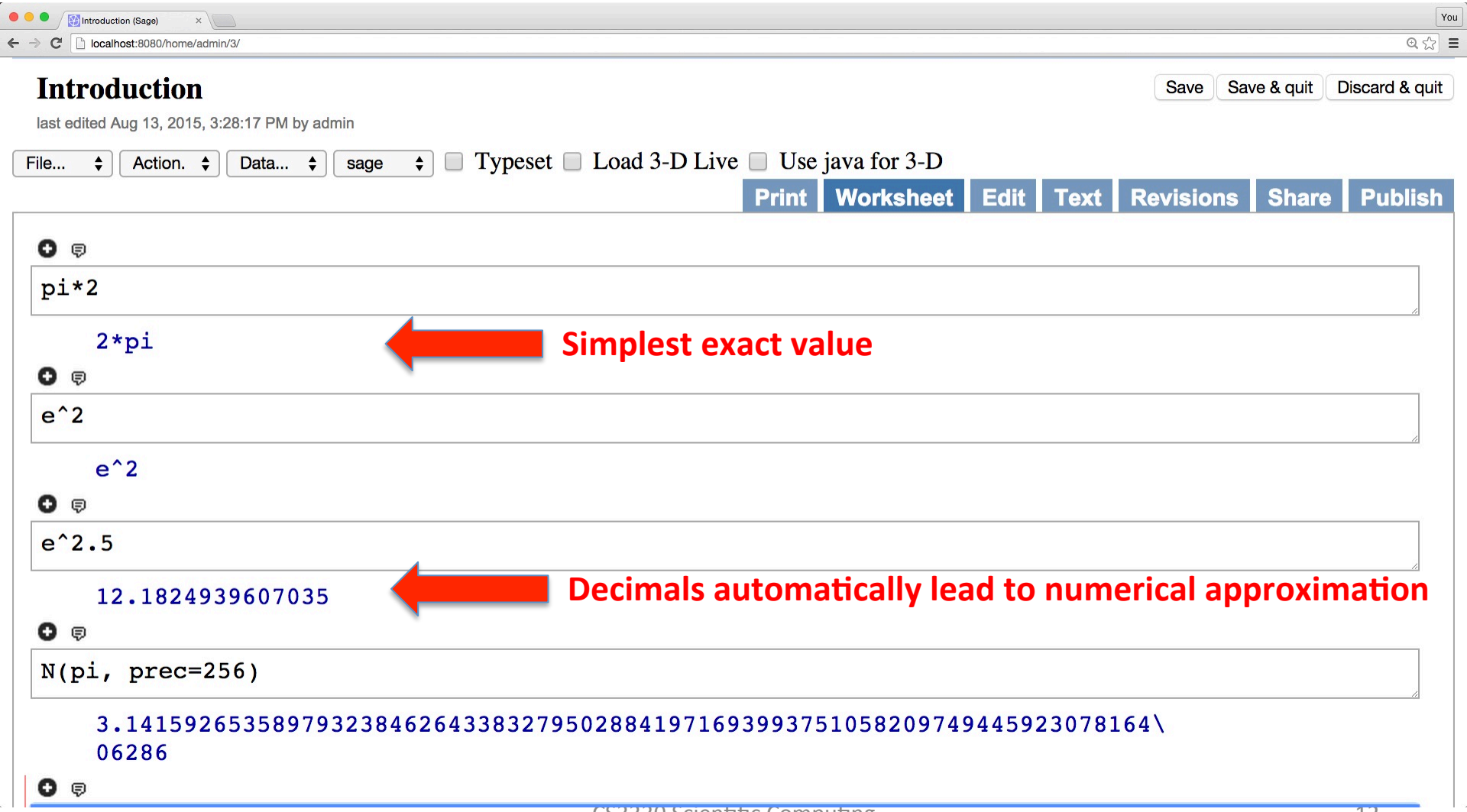
Use SageMath as a Calculator



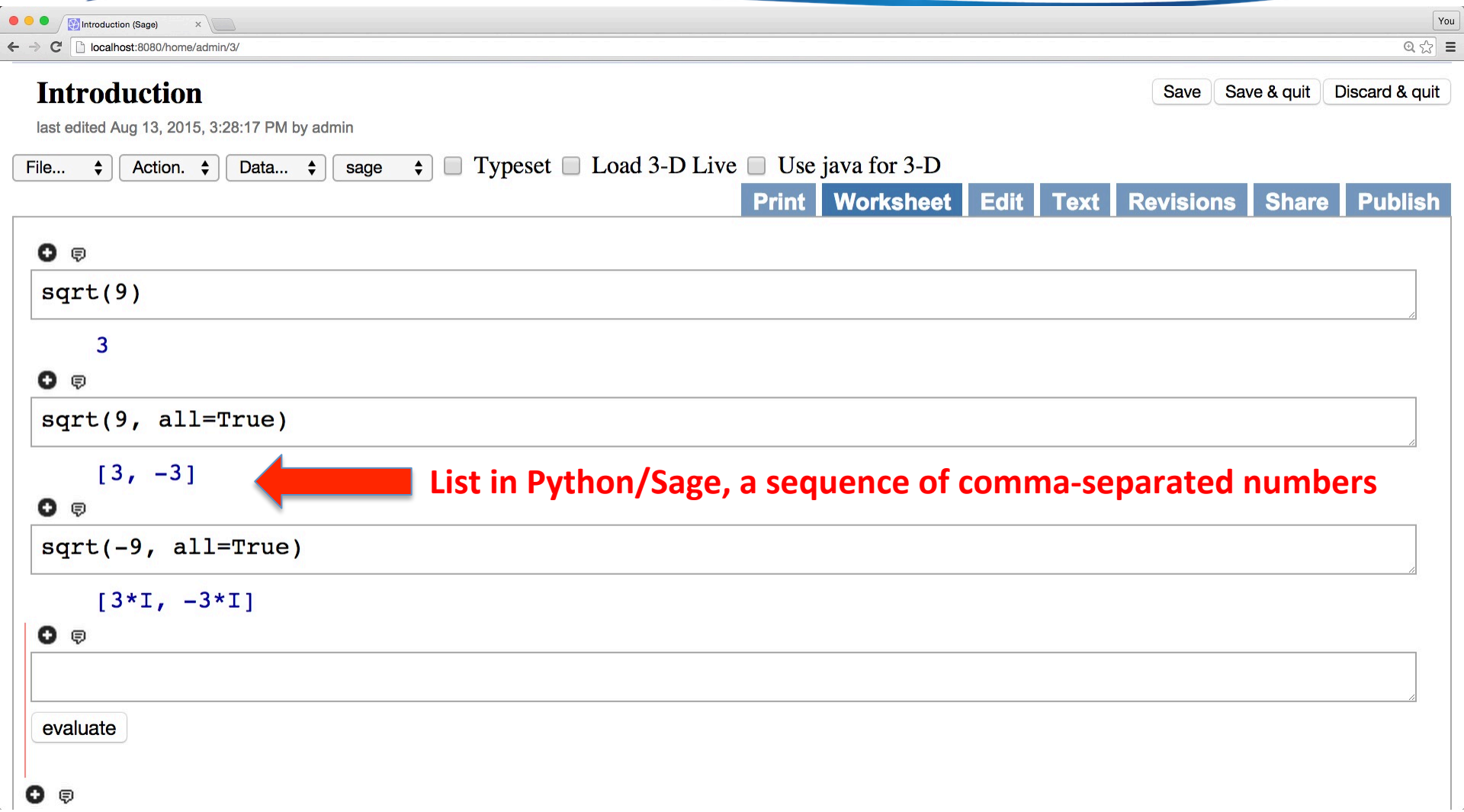
Square Root Function



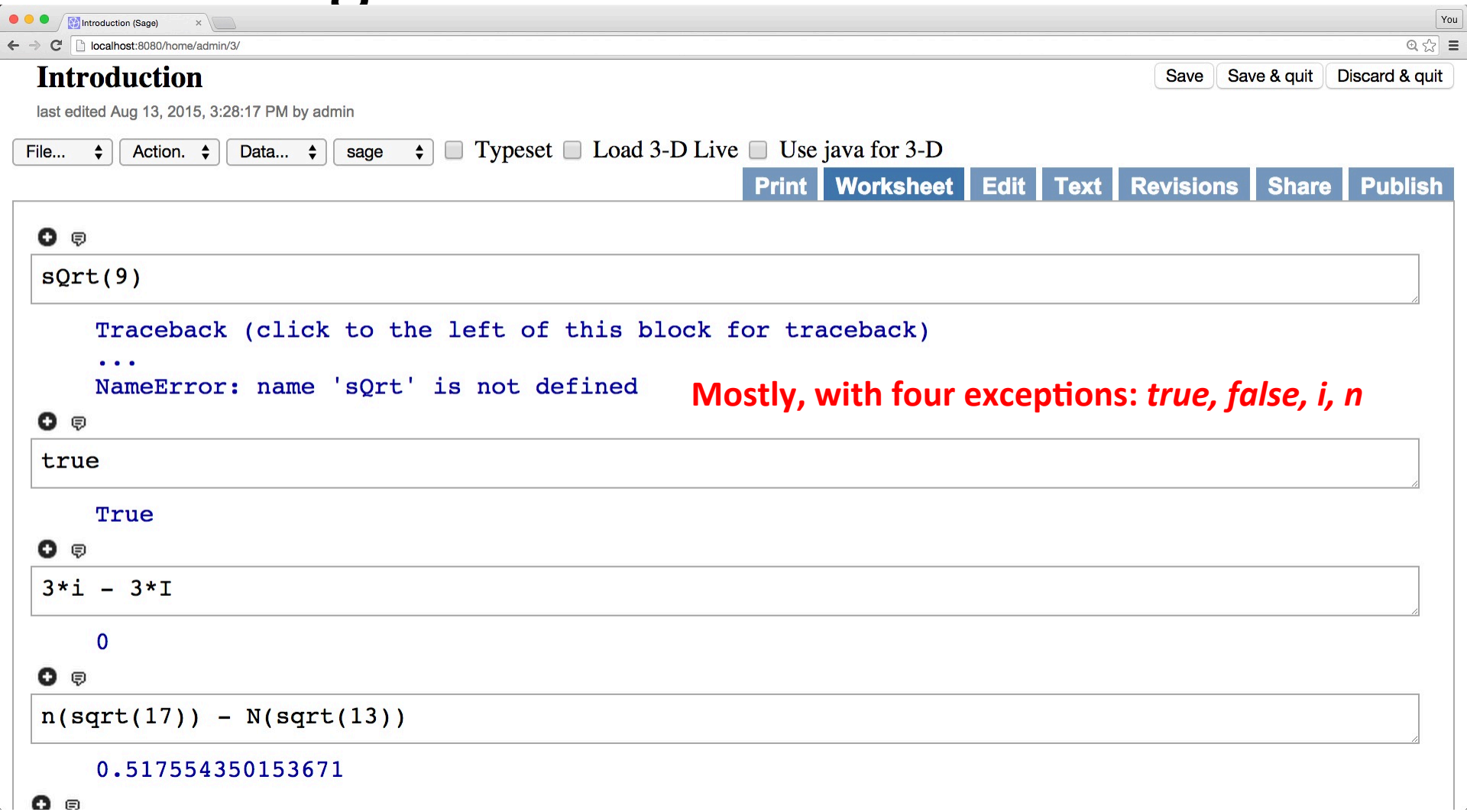
Special Constants



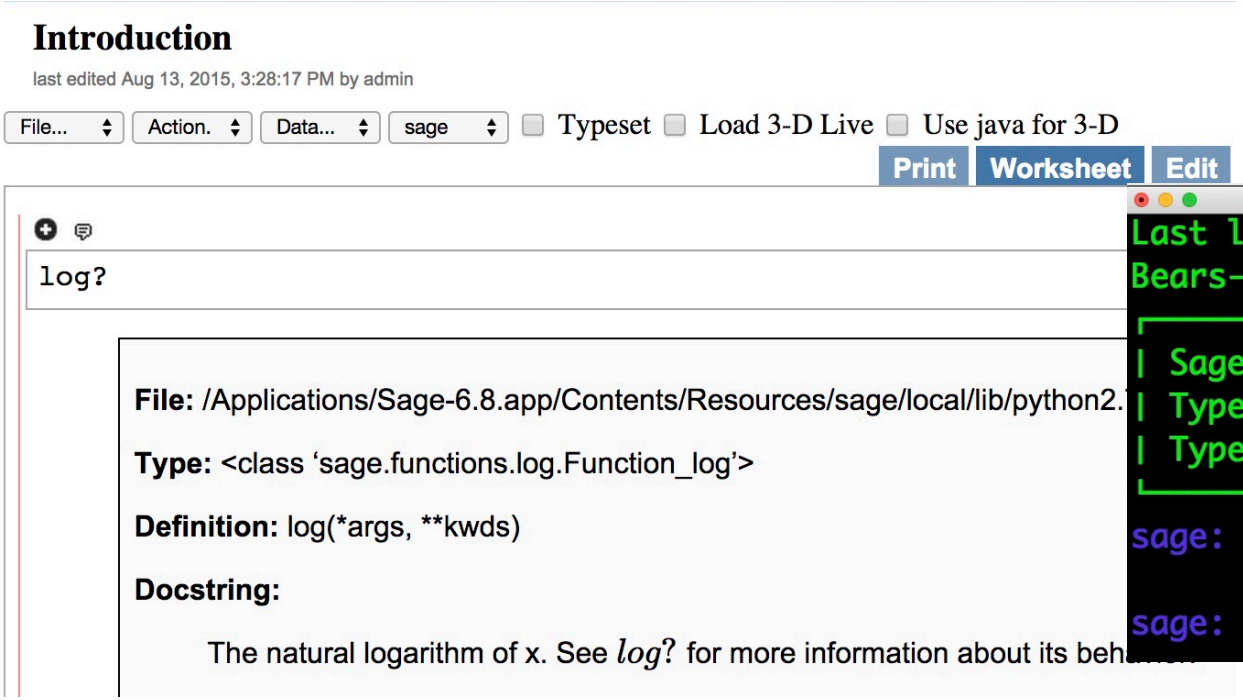
More on Square Roots

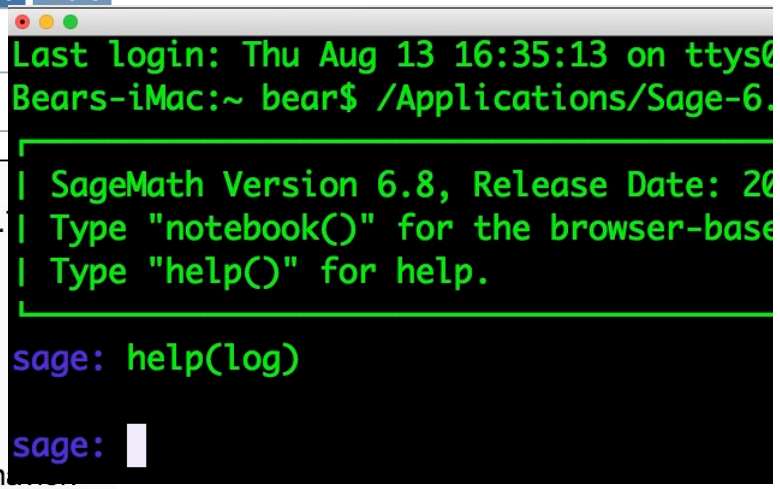


SageMath is Case Sensitive



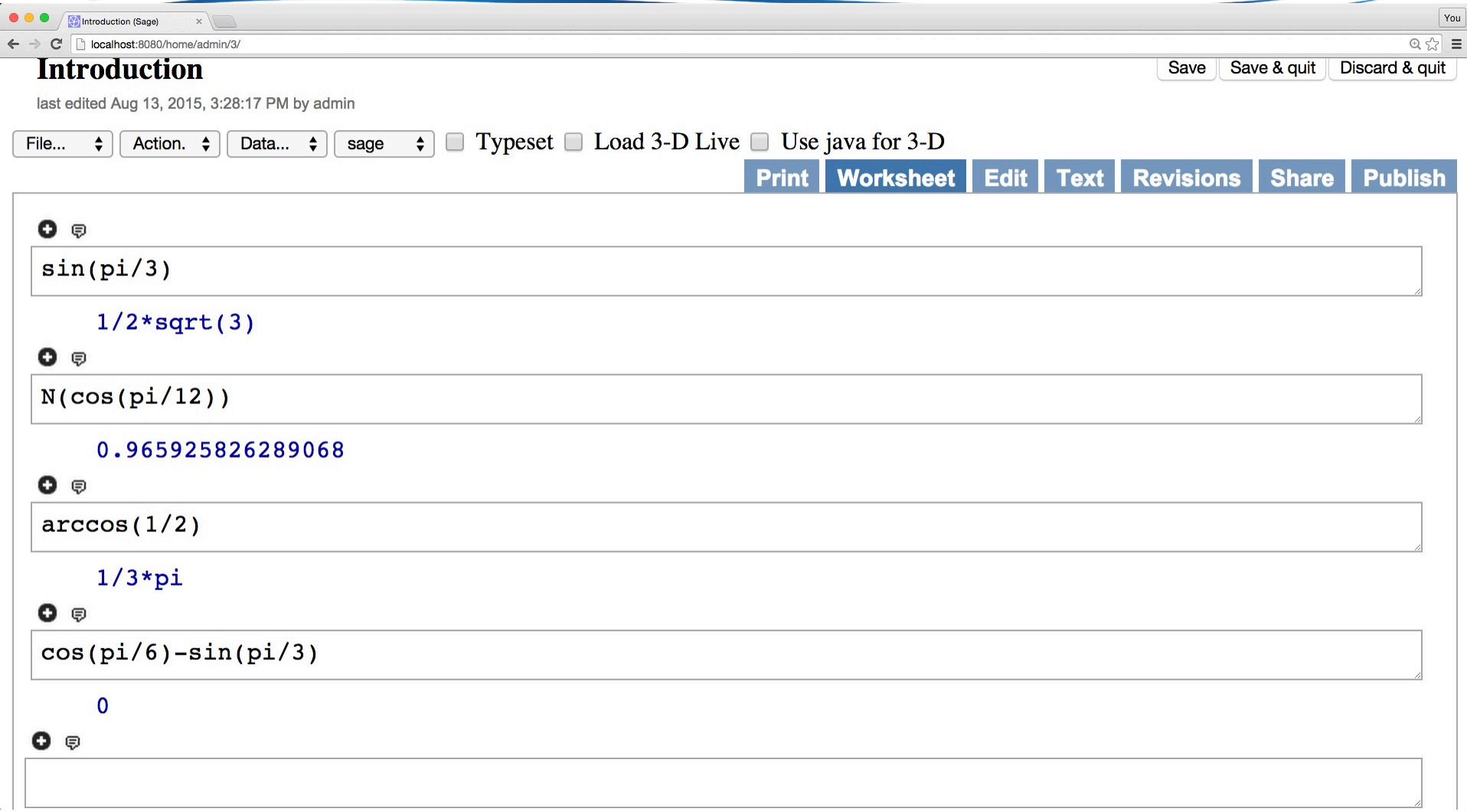
How to Get Help





- Notebook: *command*?
- Command-line: help(*command*)
- Also, utilize the *Tab-completions*

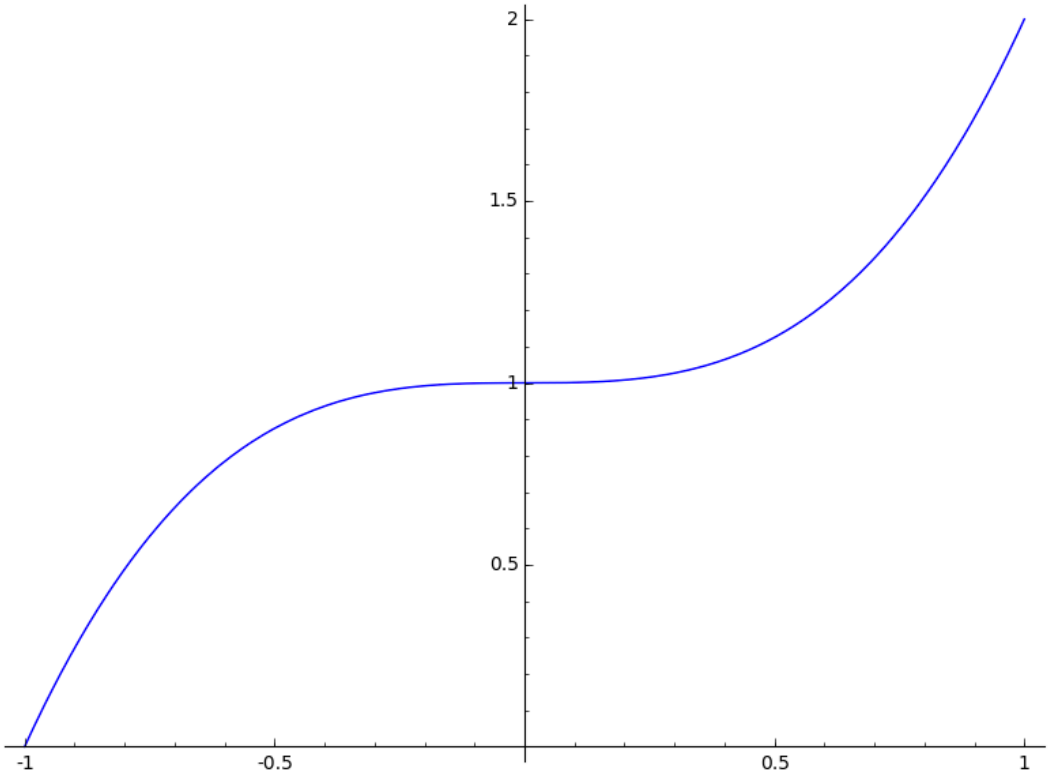
Trigonometry



Simple 2D Plots

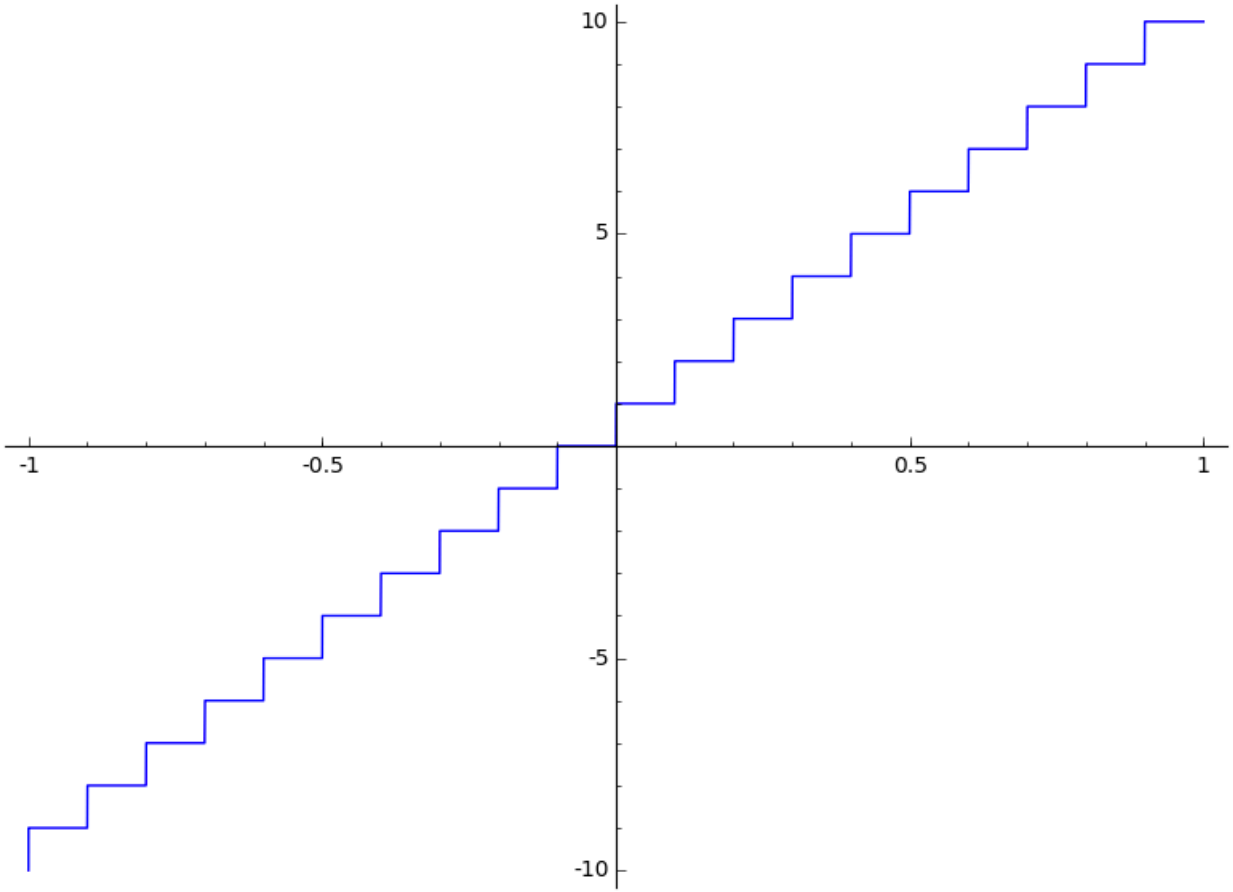
```
plot(x^3+1)
```

evaluate



More Complicated Function

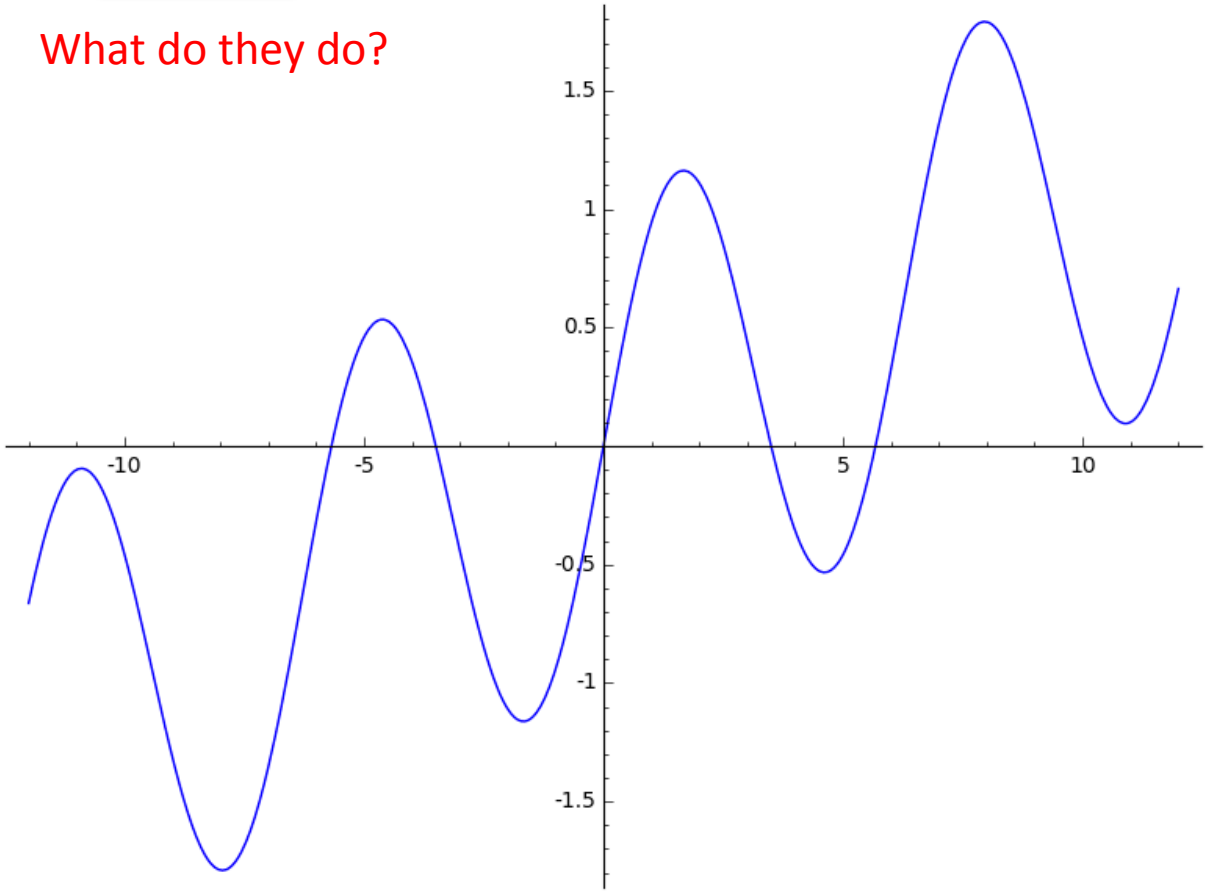
```
plot(ceil(x*10))
```



Even More Complicated

```
plot(sin(x)+0.1*x, -12, 12)
```

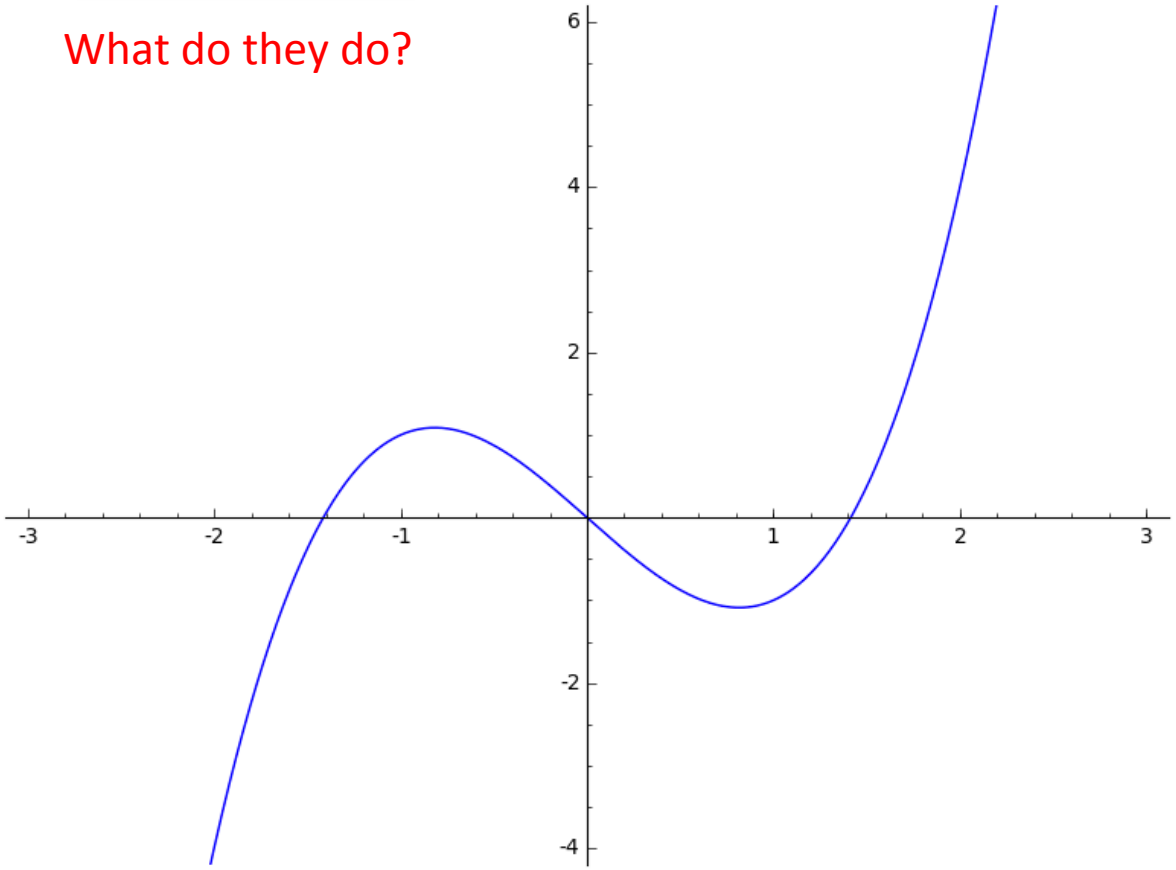
What do they do?



Ranges of Axes

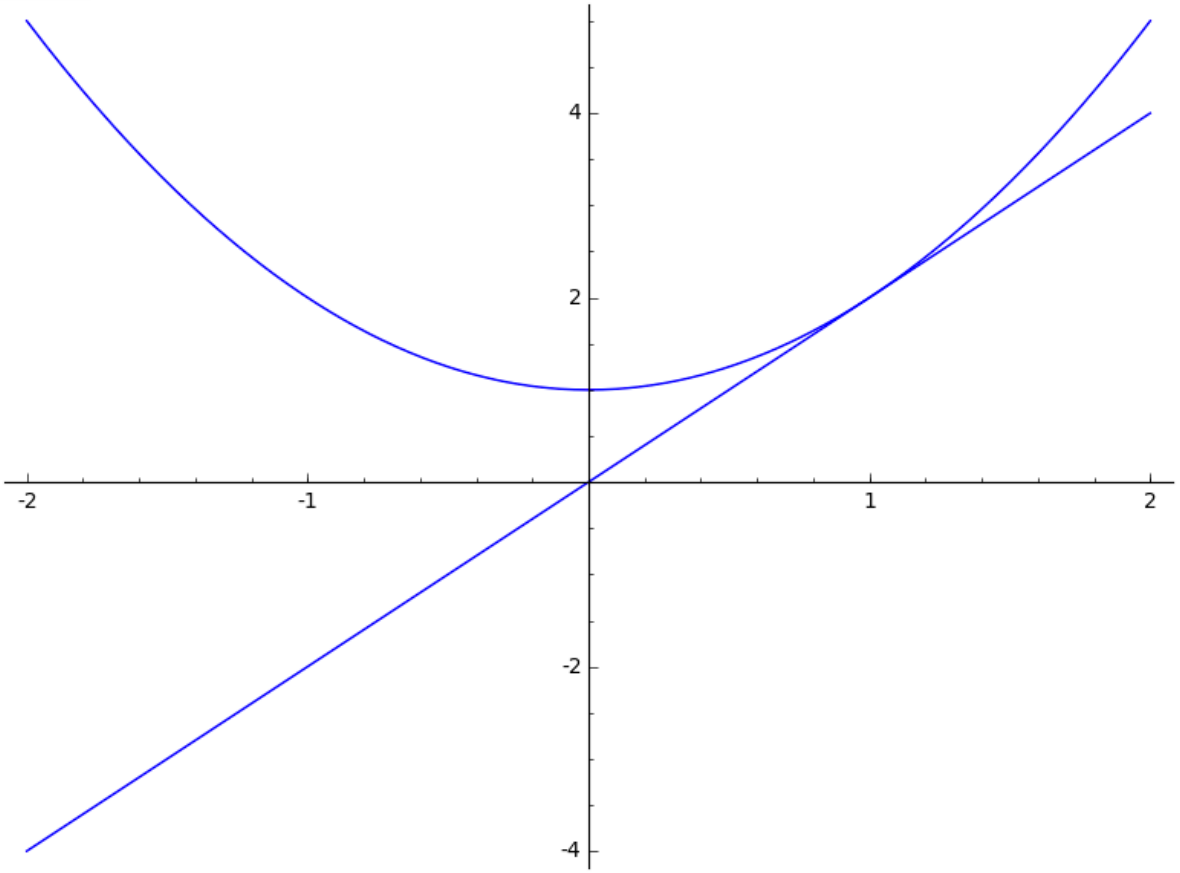
```
plot(x^3-2*x, -3, 3, ymin=-4, ymax=6
```

What do they do?



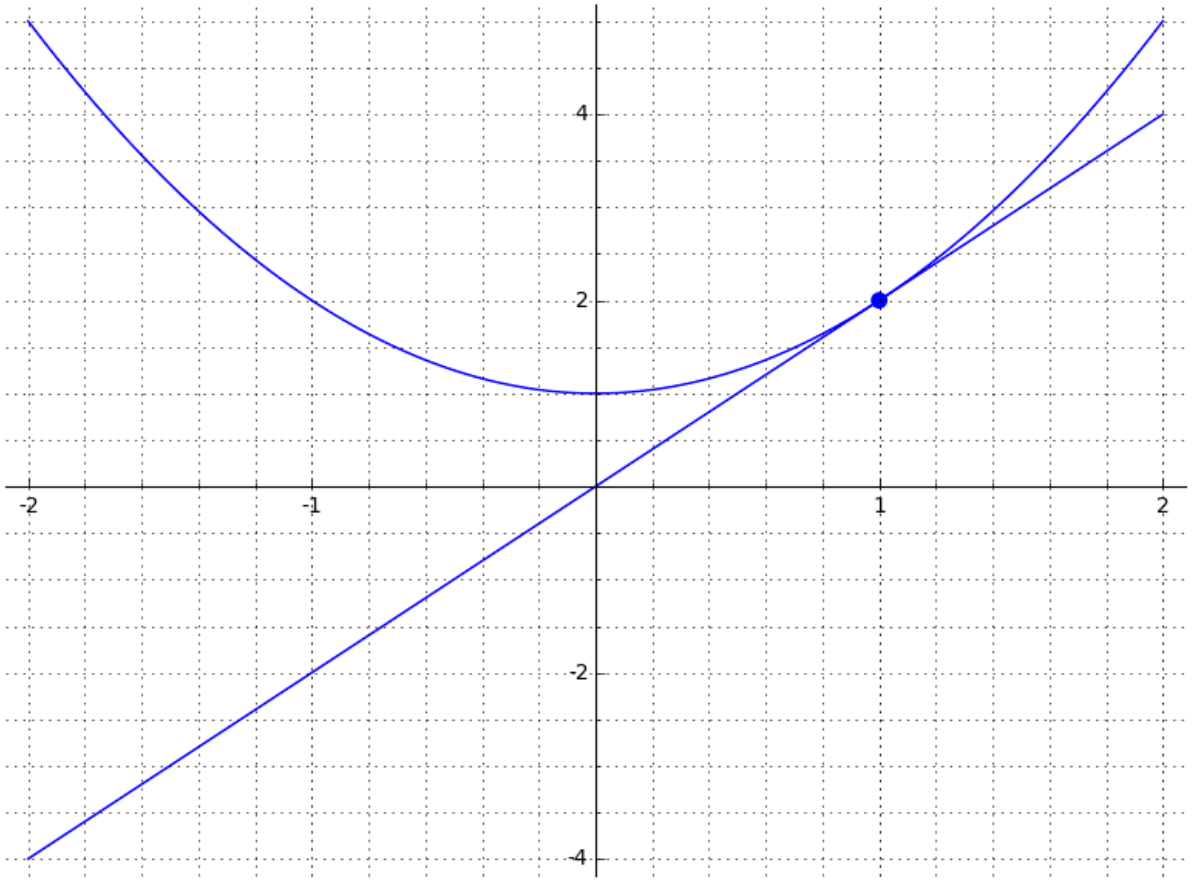
Superimposing of Multiple Graph

```
plot(2*x, -2, 2) + plot (x^2+1, -2,2)
```



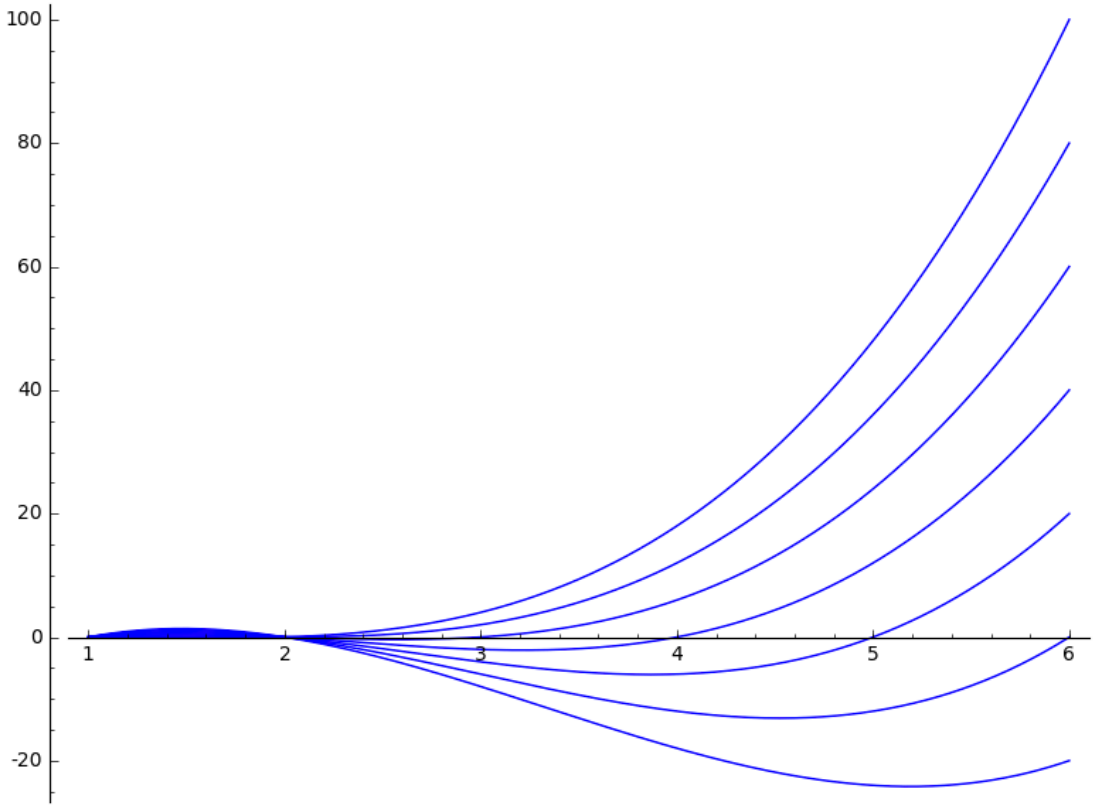
Polished Graph

```
plot(2*x, -2, 2, gridlines='minor') + plot (x^2+1, -2,2) + point ((1,2), size=60)
```



Last Example on Graphs

```
plot((x-1)*(x-2)*(x-7), 1,6)+plot((x-1)*(x-2)*(x-6), 1,6)+plot((x-1)*(x-2)*(x-5),
1,6)+plot((x-1)*(x-2)*(x-4), 1,6)+plot((x-1)*(x-2)*(x-3), 1,6)+plot((x-1)*(x-2)*(x-2),
1,6)+plot((x-1)*(x-2)*(x-1), 1,6)
```



Reduced Row Echelon Form (RREF)

- System of equations:

$$\begin{aligned} 3x - 4y + 5z &= 14 \\ x + y - 8z &= -5 \\ 2x + y + z &= 7 \end{aligned}$$

$$A = \left[\begin{array}{ccc|c} 3 & -4 & 5 & 14 \\ 1 & 1 & -8 & -5 \\ 2 & 1 & 1 & 7 \end{array}\right]$$

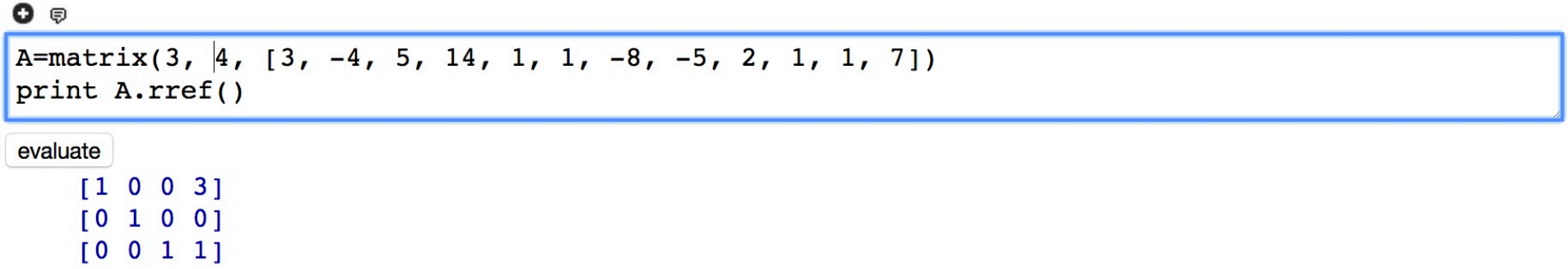
- RREF:

$$A' = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$x = 3, y = 0, z = 1$$

Getting RREF in SageMath

- Create a matrix ← what are the first two arguments of *matrix(.)*?
- Invoke *rref(.)* function



Practice, a Tricky Question

- Try to solve the following equation system using SageMath

$$2x - 5z + y = 6 + w$$
$$5 + z - y = 0$$
$$w + 3(x + y) = z$$
$$1 + 2x - y = w - 3x$$

- Answer:

$$w = \frac{-107}{7}, x = \frac{-12}{7}, y = \frac{54}{7}, z = \frac{19}{7}$$

Initializing Matrices

- Two ways to create matrices

```
B = matrix(2, 3, [1, 2, 3, 4, 5, 6])
print B
```

```
[1 2 3]
[4 5 6]
```

```
B = matrix([ [1, 2, 3], [4, 5, 6] ])
print B
```

```
[1 2 3]
[4 5 6]
```

Exceptional RREF

$$\begin{aligned} x + 2y + 3z &= 7 \\ 4x + 5y + 6z &= 16 \\ 7x + 8y + 9z &= 24 \end{aligned}$$

$$C1 = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 4 & 5 & 6 & 16 \\ 7 & 8 & 9 & 24 \end{array}\right]$$

```
B = matrix(3, 4, [1, 2, 3, 7, 4, 5, 6, 16, 7, 8, 9, 24])
print B.rref()
```

```
[ 1  0 -1  0]
[ 0  1  2  0]
[ 0  0  0  1]
```

- What is the solution?

Exceptional RREF (cont.)

$$\begin{aligned} x + 2y + 3z &= 7 \\ 4x + 5y + 6z &= 16 \\ 7x + 8y + 9z &= 25 \end{aligned}$$

$$C2 = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 4 & 5 & 6 & 16 \\ 7 & 8 & 9 & 25 \end{array}\right]$$

```
B = matrix(3, 4, [1, 2, 3, 7, 4, 5, 6, 16, 7, 8, 9, 25])
print B.rref()
```

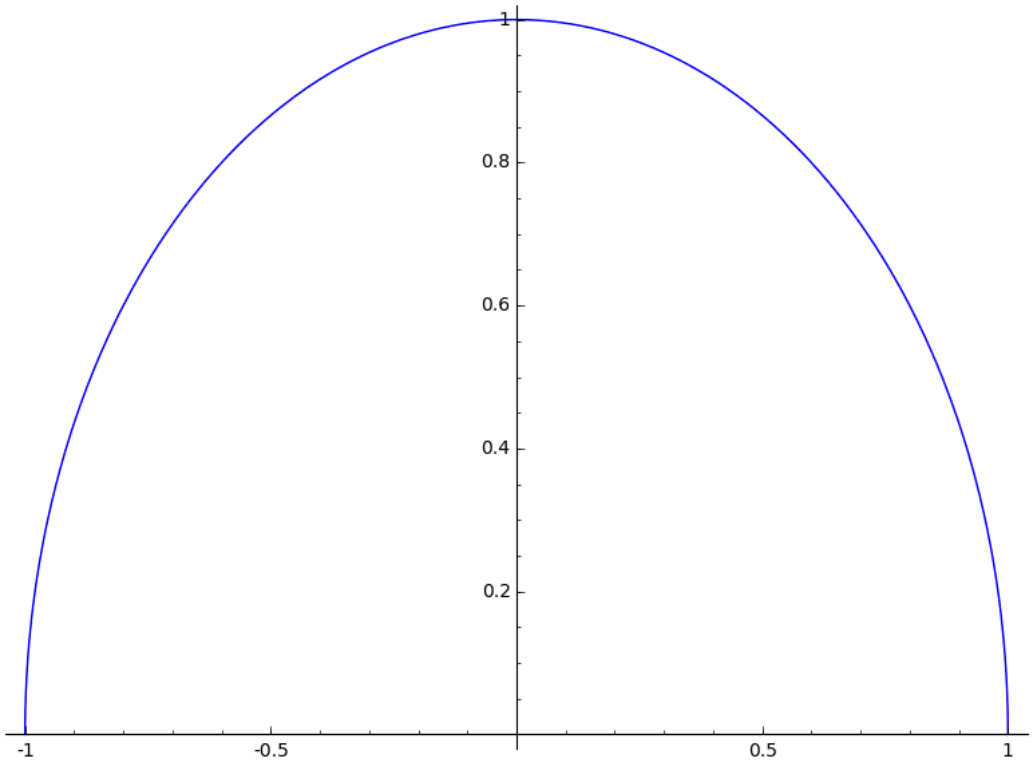
```
[ 1  0 -1 -1]
[ 0  1  2  4]
[ 0  0  0  0]
```

- What are the solutions?

Define our Own Functions

```
f(x)=sqrt(1-x^2)
print(f(0.1))
print(f(0.001))
print(f(0.00001))
print(f(0.0000001))
print(f(0.000000001))
plot(f, -1, 1)
```

```
0.994987437106620
0.999999499999875
0.999999999950000
0.999999999999995
1.00000000000000
```



Function Composition

```
f(x) = 3*x+5
g(x) = x^3 + 1
h(x) = f(g(x))
print h
h(-2)
```

```
x |--> 3*x^3 + 8
-16
```

```
f(x) = 3*x+5
g(x) = x^3 + 1
h(x) = f(g(x))
print h
h2(x) = g(f(x))
print h2
```

```
x |--> 3*x^3 + 8
x |--> (3*x + 5)^3 + 1
```

Factorization

```
print factor(12345)
print factor(987654321)
print factor(x^2 - 8*x + 15)
print gcd(x^2 - 5*x +6, x^2 - 8*x + 15)
```

```
3 * 5 * 823
3^2 * 17^2 * 379721
(x - 3)*(x - 5)
x - 3
```

Expanding a Function

```
a(x) = x^2 - 5 *x + 6
b(x) = x^2 - 8 *x + 15
f(x) = a(b(x))
print f(x)
print f.expand()
print f.factor()
```

```
(x^2 - 8*x + 15)^2 - 5*x^2 + 40*x - 69
x |--> x^4 - 16*x^3 + 89*x^2 - 200*x + 156
(x^2 - 8*x + 13)*(x - 2)*(x - 6)
```

Use Sage to Solve Problem Symbolically

- Solving $\frac{x^2}{2} - x - 2 = 0$
  - Symbolically, we get

We will start from here

$$1 \pm \sqrt{5}$$

  - Numerically, we get

$$3.23606797749979 \text{ and } -1.23606797749979$$

Single-Variable Formulas

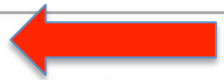
```
solve(x^2 + 3*x + 2, x)
```

```
[x == -2, x == -1]
```

```
solve(x^2 + 2*x + 3, x)
```

```
[x == -I*sqrt(2) - 1, x == I*sqrt(2) - 1]
```

```
var('alpha')
solve(cos(alpha) == 1/2, alpha)
```



Declare variable, notice that x is always a variable without explicit declaration

```
[alpha == 1/3*pi]
```

Multiple-Variable Formulas

```
var('a b c')
solve(a*x^2 + b*x + c == 0, x)
```

```
[x == -1/2*(b + sqrt(b^2 - 4*a*c))/a, x == -1/2*(b - sqrt(b^2 - 4*a*c))/a]
```

```
var('b')
solve([x+b ==6, x-b == 4], x, b)
```

```
[[x == 5, b == 1]]
```

```
var('a, b, c')
solve([9*a + 3*b +c == 32, 4*a + 2*b +c == 15, a + b + c ==6], a, b, c)
```

```
[[a == 4, b == -3, c == 5]]
```

What does it do? Remember how to solve the same problem using RREF?

Non-Linear Equation Systems

```
solve ((x-1) * (x-2) * (x-3) * (x-4) * (x-5) == 0, x)
```

```
[x == 1, x == 2, x == 3, x == 4, x == 5]
```

```
answers = solve ((x-1) * (x-2) * (x-3) * (x-4) * (x-5) == 0, x)
print answers[0]
print answers[3]
```

```
x == 1
x == 4
```

```
var('p q y')
eq1 = p+q == 9
eq2 = q*y + p*x == -6
eq3 = q*y^2 + p*x^2 == 24
eq4 = p == 2
solve([eq1, eq2, eq3, eq4], p, q, x, y)
```

List of list? How to make it easier to read?

evaluate

```
[[p == 2, q == 7, x == -1/3*sqrt(70) - 2/3, y ==
2/21*sqrt(7)*sqrt(5)*sqrt(2) - 2/3], [p == 2, q == 7, x == 1/3*sqrt(70)
- 2/3, y == -2/21*sqrt(7)*sqrt(5)*sqrt(2) - 2/3]]
```

Higher Order Equations

```
answers = solve([x^2 - y^2 ==1, (x^2)/4 + (y^2)/3 == 1], x, y)
print answers[0]
print answers[1]
print answers[2]
print answers[3]
```

```
[x == -4/7*sqrt(7), y == -3/7*sqrt(7)]
[x == -4/7*sqrt(7), y == 3/7*sqrt(7)]
[x == 4/7*sqrt(7), y == -3/7*sqrt(7)]
[x == 4/7*sqrt(7), y == 3/7*sqrt(7)]
```

```
solve(log( x^2 ) == 5/3, x)
```

evaluate

```
[x == -e^(5/6), x == e^(5/6)]
```

```
solve(sin(x+y)==0.5, x)
```

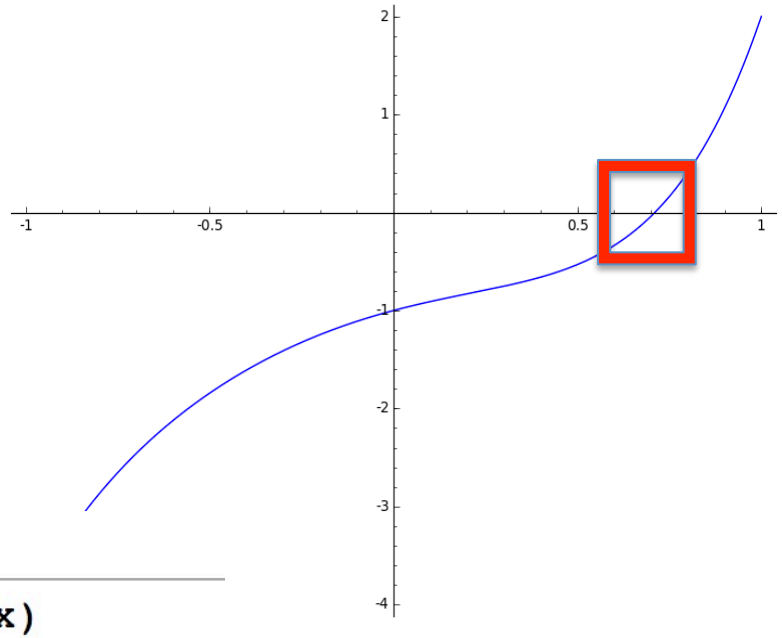
Try this to see what we get

```
[x == 1/6*pi - y]
```

```
var('a0, a1, a2, a3, a4')
solve(a4*x^4 + a3*x^3 + a2*x^2 + a1*x + a0 == 0, x)
```

Quantic Polynomials and Numerical Solver

Solve $x^5 + x^4 + x^3 - x^2 + x - 1 = 0$




```
solve(x^5 + x^4 + x^3 - x^2 + x - 1 == 0, x)
```

```
[0 == x^5 + x^4 + x^3 - x^2 + x - 1]
```

Why can't we find a solution?

```
find_root(x^5 + x^4 + x^3 - x^2 + x - 1, -100, 100)
```

```
0.7104342557869131
```

More Numerical Solver Examples

```
find_root(e^x - 1/x, -1, 1)
```

```
0.5671432904097954
```

```
find_root(cos(x)==sin(x), 0, pi/2)
```

evaluate

```
0.7853981633974484
```

```
var('n')
find_root(7000 == 5000*(1+0.045/12)^n, 0, 100000)
```

```
89.89406093308004
```

Exercise: what are we calculating?

How to Take Derivatives

```
diff(x^4 - x^2 + x , x)
```

```
4*x^3 - 2*x + 1
```

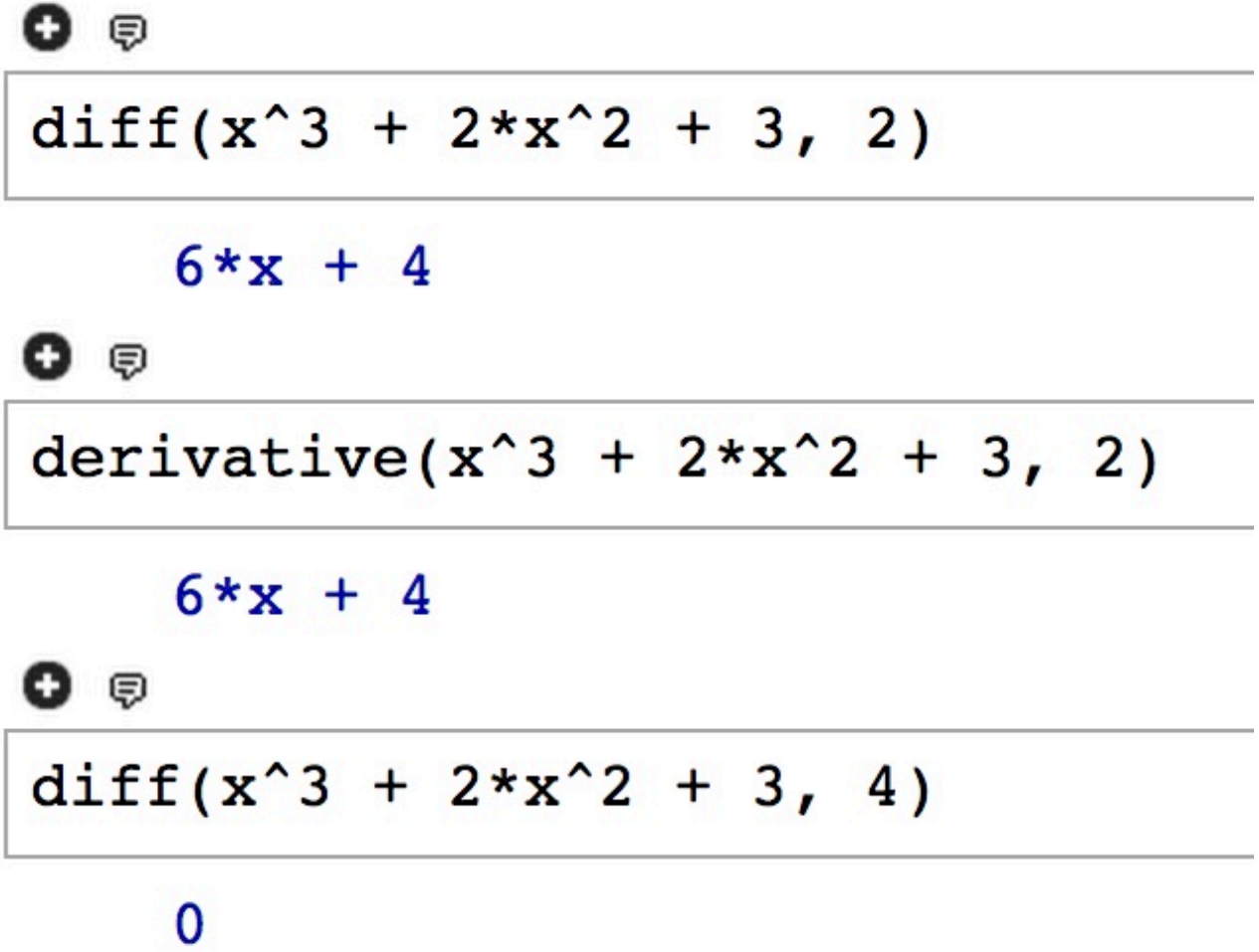
```
diff(sin(x/e), x)
```

```
cos(x*e^(-1))*e^(-1)
```

```
g(x)=e^(-10*x)
g.derivative()
f(x) = g.derivative()
f(-12)
```

```
-10*e^120
```

Higher Order Derivatives



Indefinite Integral

```
integral(x*cos(x^2), x)
```

```
1/2*sin(x^2)
```

$$\int x\cos(x^2)dx = \frac{1}{2}sin(x^2) \quad ?$$

```
integral(x / (x^2 + 2), x)
```

```
1/2*log(x^2 + 2)
```

$$\int \frac{x}{x^2+2}dx = \frac{1}{2}\log(x^2+2) \quad ?$$

Don’t forget the constant!

Definite and Impossible Integrals

```
integral(x*cos(x^2), x, -2, 3)
```

```
1/2*sin(9) - 1/2*sin(4)
```

$$\int_{-2}^{3} x\cos(x^2)dx$$

```
integral(x / (x^2 + 2), x, 1, 2)
```

```
1/2*log(6) - 1/2*log(3)
```

$$\int_{1}^{2} \frac{x}{x^2+2}dx$$

```
var('y')
assume(y>0)
integral(2 / sqrt(pi) * (e ^ (-x^2)), x, 0, y)
```

```
erf(y)
```

We even define a special function for this!

Numerical Integrations and Partial Fractions

```
integral(exp(-x^3) * sin(x^2), x, 1, 3)
```

```
integrate(e^(-x^3)*sin(x^2), x, 1, 3)
```

```
numerical_integral(x / (x^2 + 2), 1, 2)
```

```
(0.3465735902799726, 3.847739796558e-15)
```

```
f(x) = (x^3 - x) / (x^2 + 5*x + 6)
f.partial_fraction()
```

```
x |--> x + 24/(x + 3) - 6/(x + 2) - 5
```

Summary

- We introduced SageMath, an opensource project based on Python
- We go over basics of SageMath, including symbolic and numerical solutions, matrices, and simple plots
- We will mostly use SageMath for symbolic solutions
  - SageMath was initially designed for this
  - Other tasks are done in Matlab/Octave
- References:
  - http://www.sagemath.org ← Official Web and resources
  - http://www.gregorybard.com/SAGE.html ← Our textbook

SageMath #1 Homework (S1)

1. (1%) Find at least 5 real number solutions of $e^{-x^2}\sin(8x) = 0$ in *[-3, 3]* using SageMath
   - Hint: Plot the curve first!
2. (1%) Run *A = random_matrix(ZZ, 3, 4)* in SageMath. Map the matrix into an equation system (say variables are *x*, *y*, and z*)*. Write down the equation system. Solve the equation system using SageMath.
3. (1%) Execute the following code in SageMath. Explain why the two prints give different answers? You need to identify the key difference to get the point.

```
x = ceil(random() * 999)
print sqrt(x)
print N(sqrt(x))
```