SageMath 1: Using SageMath

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What is SageMath

- Sage stands for *System for Algebra and Geometry Experimentation*
- Starts from a Computer Algebra System (CAS)
- Now a full spectrum of math tools, including
  - Algebra
  - Combinatorics
  - Matrix Computations
  - Numerical Math
  - Calculus
  - Number Theory
  - Visualization
  - Many other topics

http://www.sagemath.org

How is SageMath Built

- Created in 2005 by William Stein at University of Washington
- Goal: Offer an affordable math software to solve various math problems
- Approach
  - Leverage opensource (GPL-ed) software, such as *SciPy, Maxima, and GAP*
  - Adopt the popular Python as the programming language
- Outcome: A huge set of math tools!

What SageMath Can Do

- Solving exact problems

```
integral(x*sin(x^2),x)
```

```
-1/2*cos(x^2)
```

```
integral(x*sin(x^2),x, -1, 1)
```

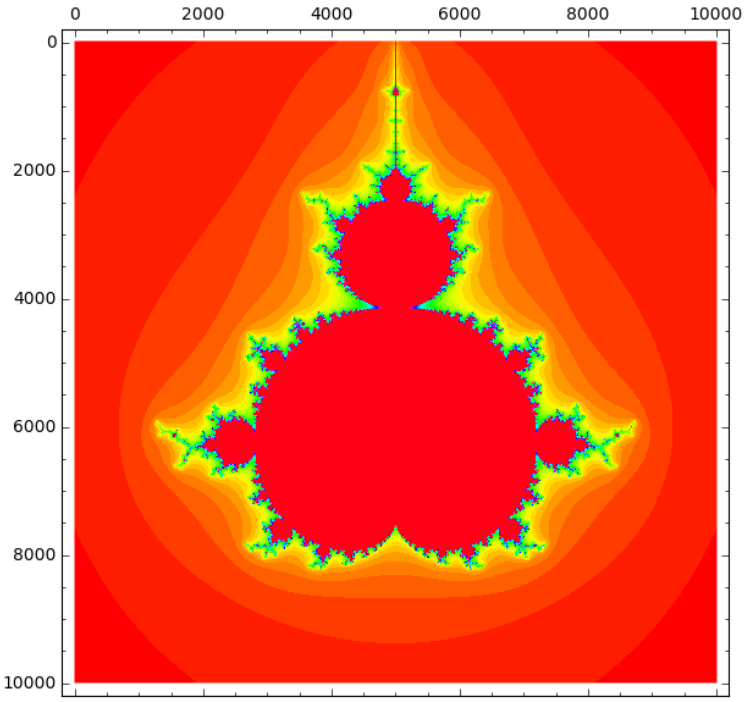
```
0
```

- Solving numerical problems

```
numerical_integral(x*sin(x^2), -1, 1)
```

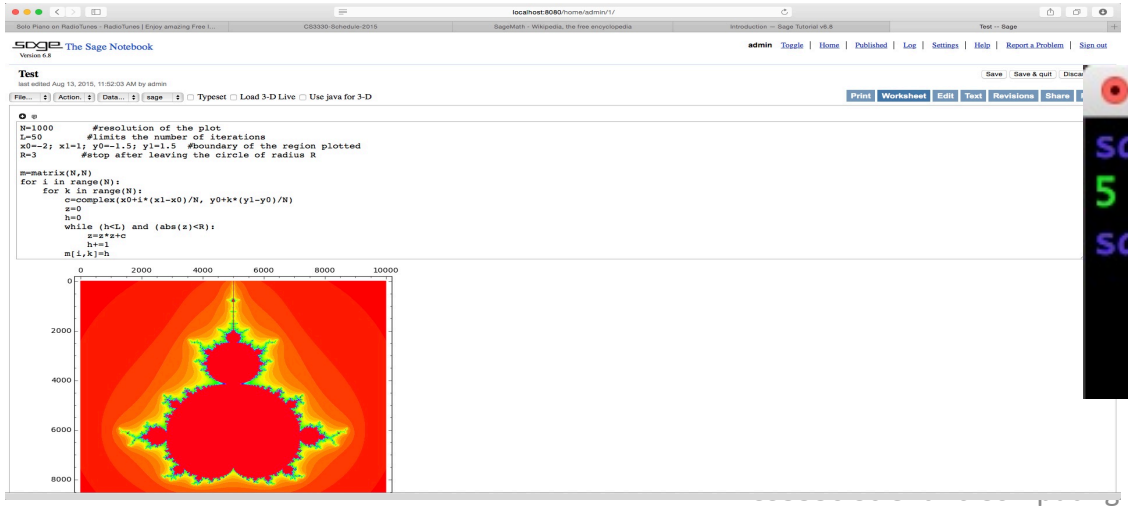
```
(0.0, 5.103670946680643e-15)
```

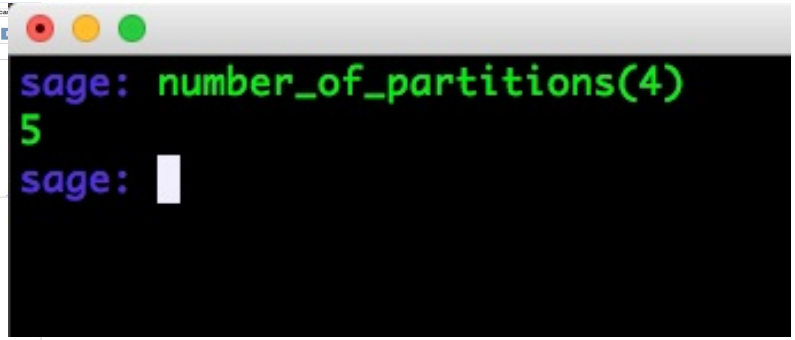
- Plot cool figures
  - Get a sense of more global behavior



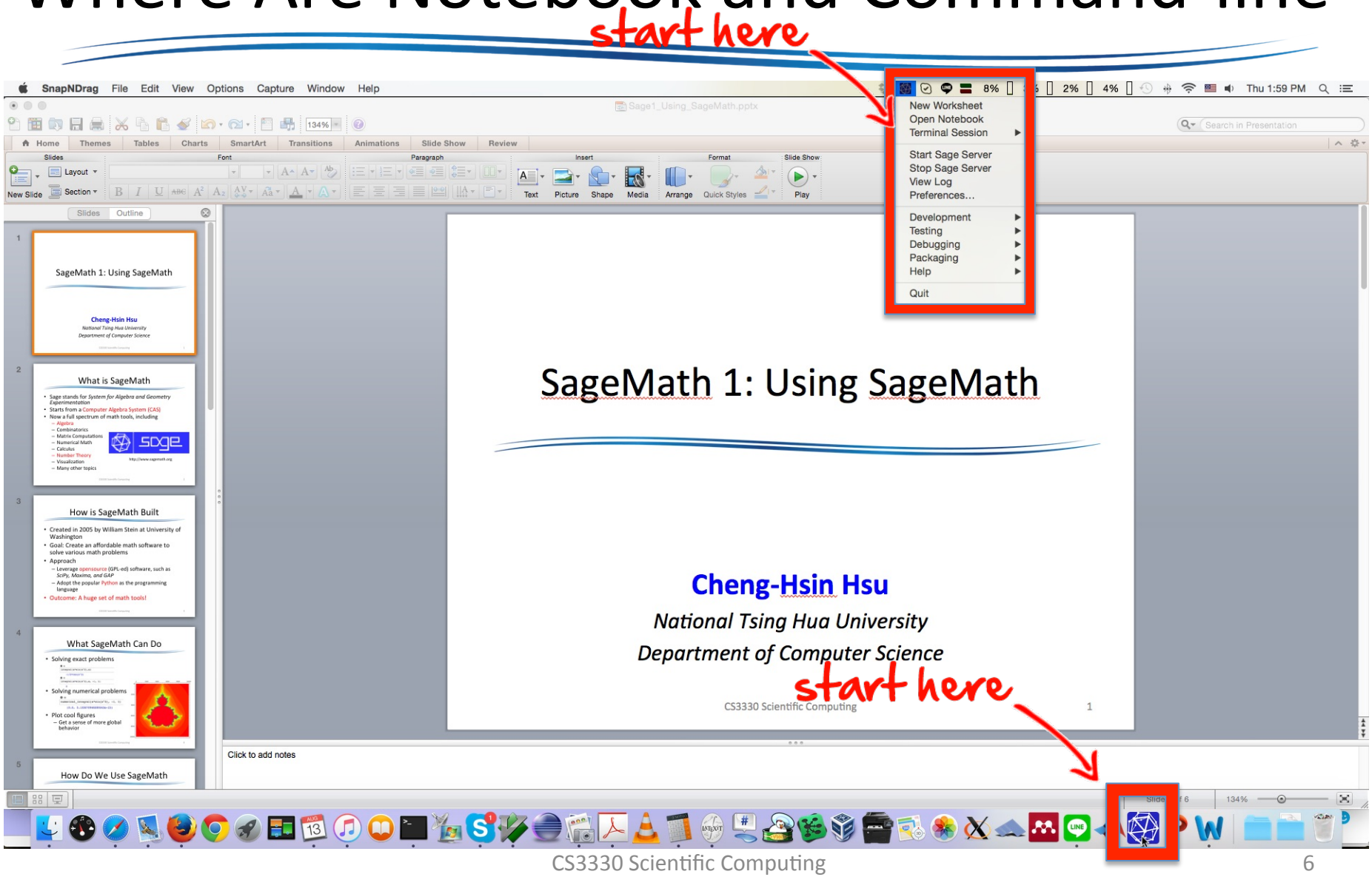
Different Ways to Use SageMath

- Notebook: Web-based interface
- SageMath Online: Cloud-based Web interface
- Interactive command-line: Python-like shell prompt
- Programs: Write and compile Sage programs
- Scripts: Invoke Sage libraries from Python

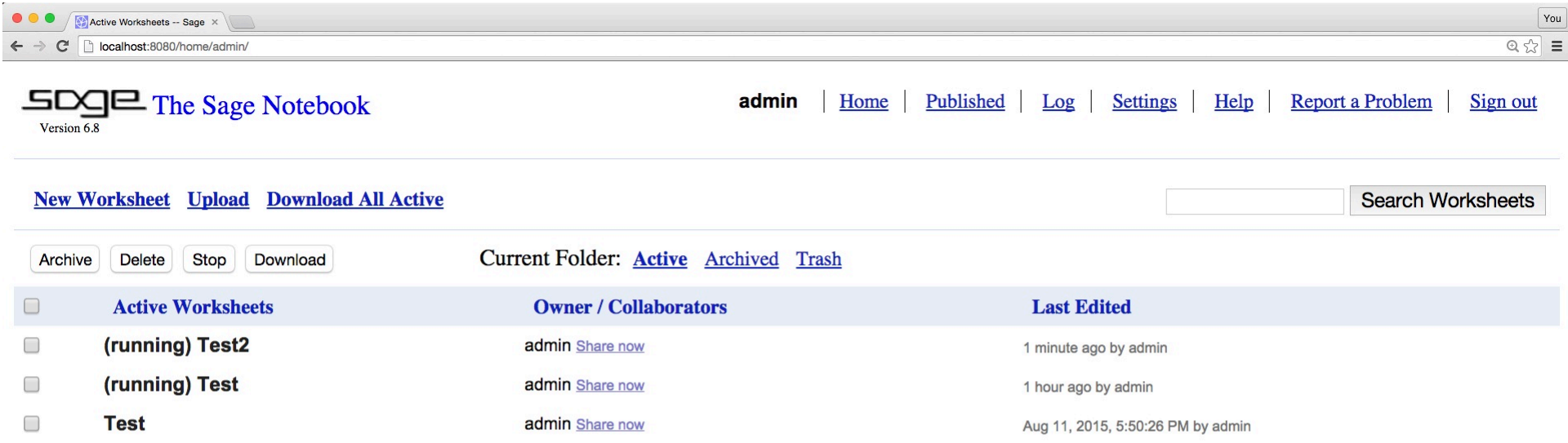




Where Are Notebook and Command-line

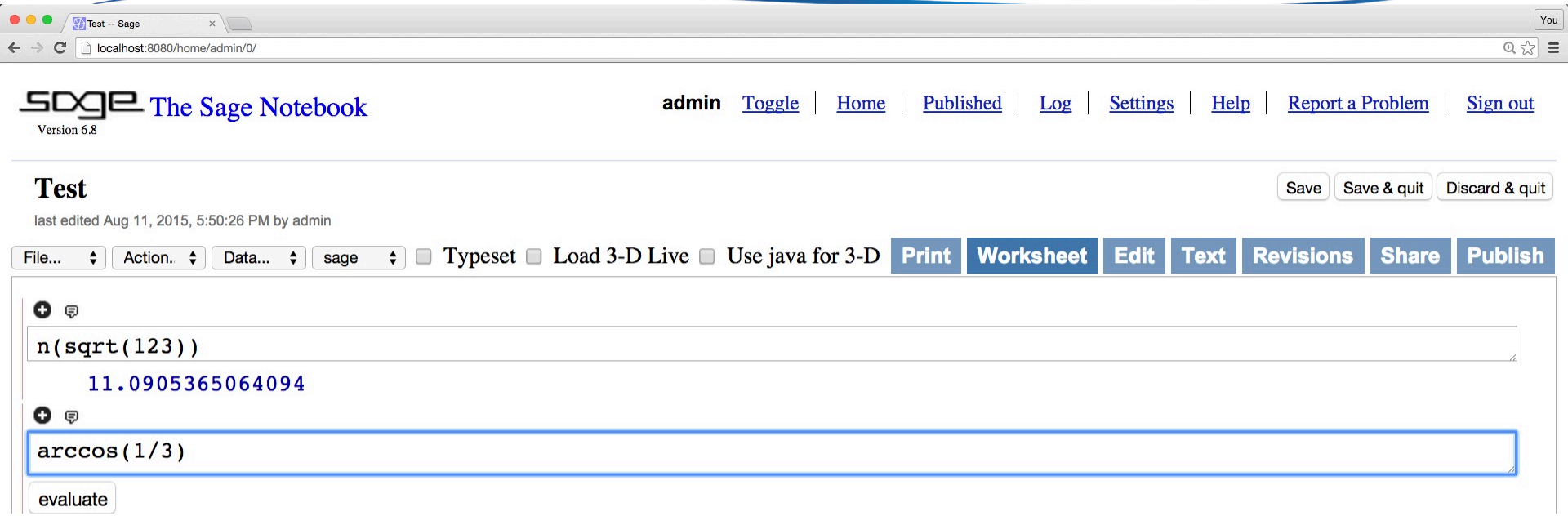


SageMath Notebook



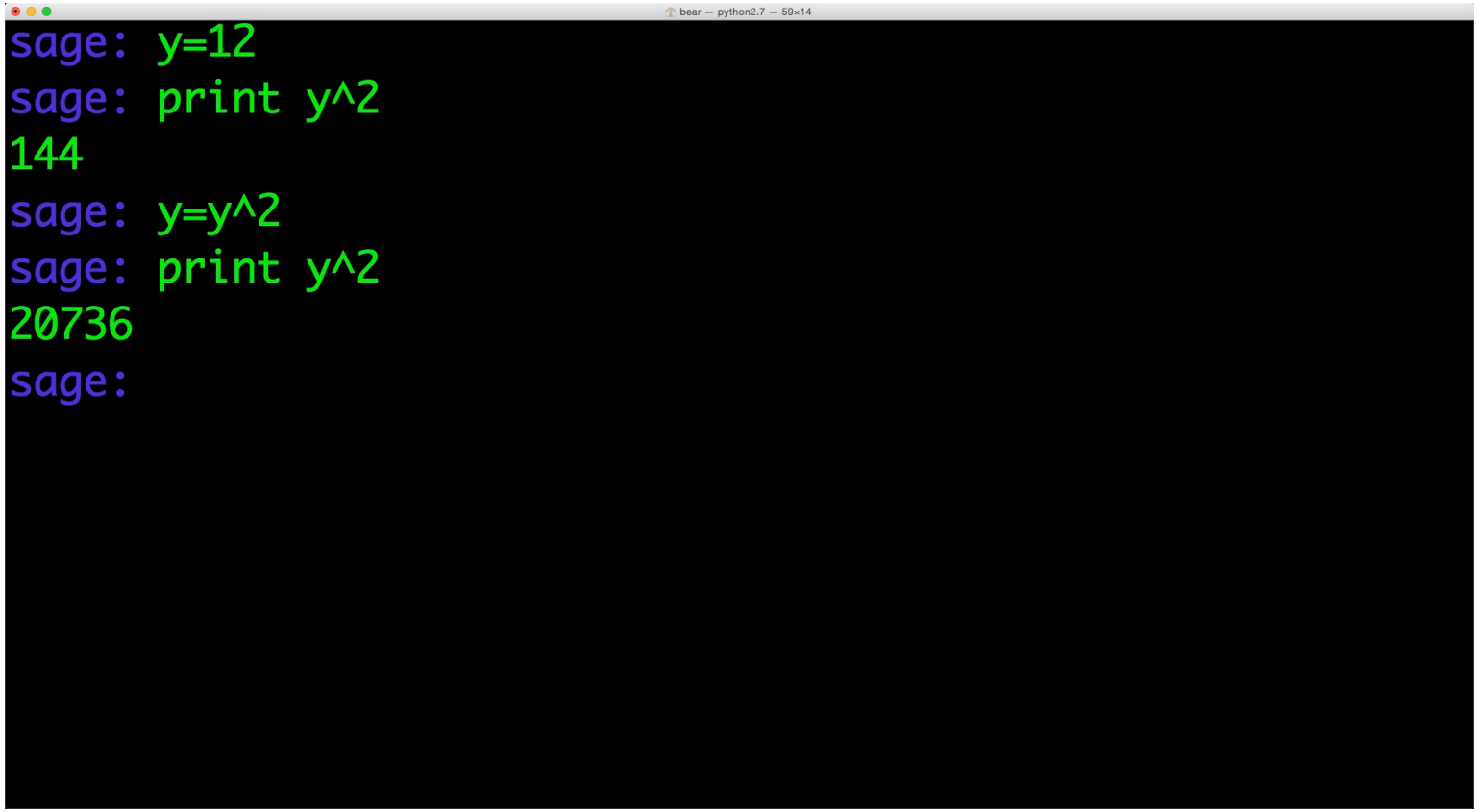
- Each notebook has several worksheets
- Worksheets can be saved, opens, downloaded, shared, and etc.
- SageMath Online: http://cloud.sagemath.com/ provides similar interface

SageMath Worksheet



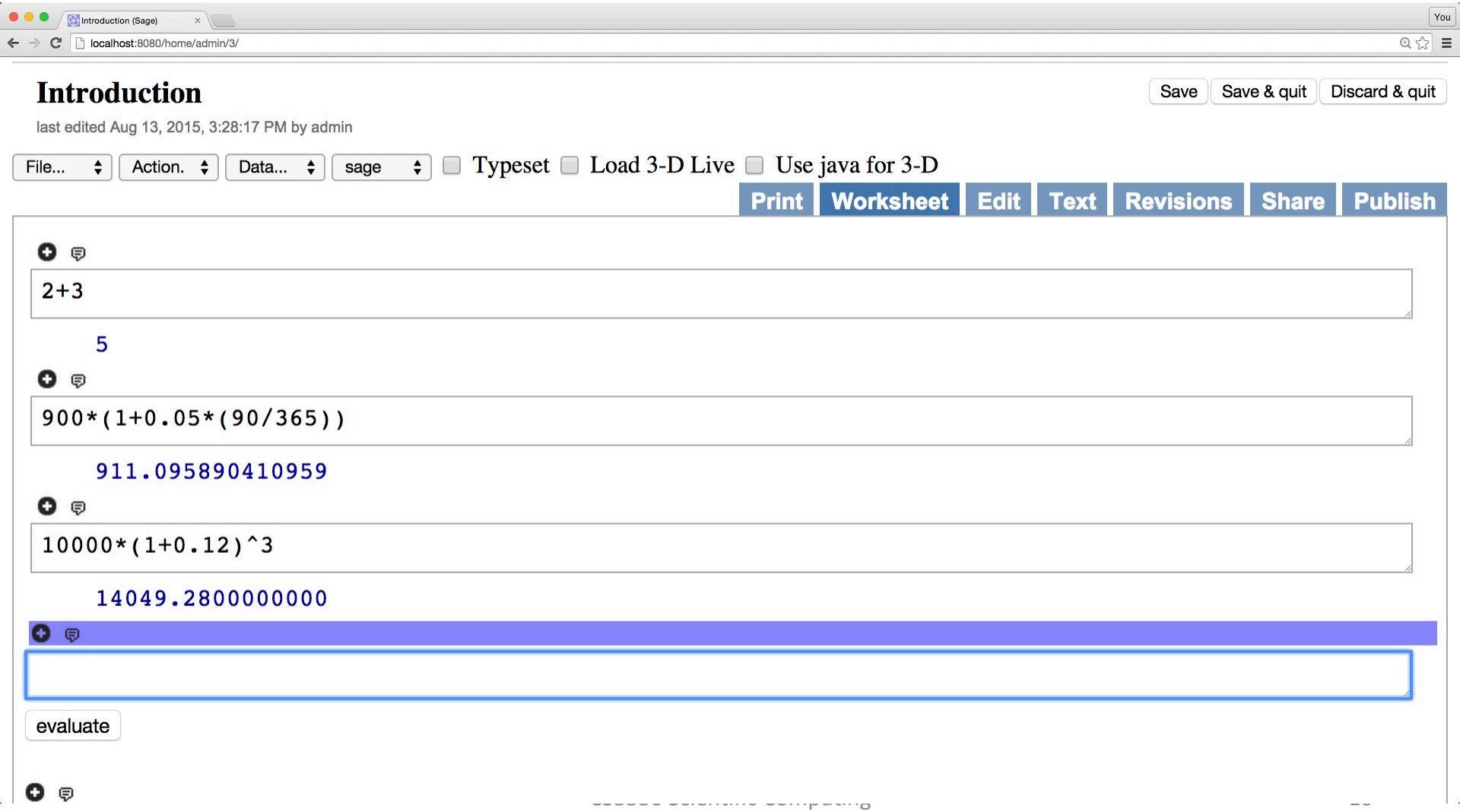
- Each worksheet contains several (independent, stateless) cells
- Each cell is a box where you can type in formulas and evaluate them ← by pressing the evaluate button or *Shift-Enter*
- You can always come back and edit your formulas/commands and re-evaluate them

SageMath Commandline

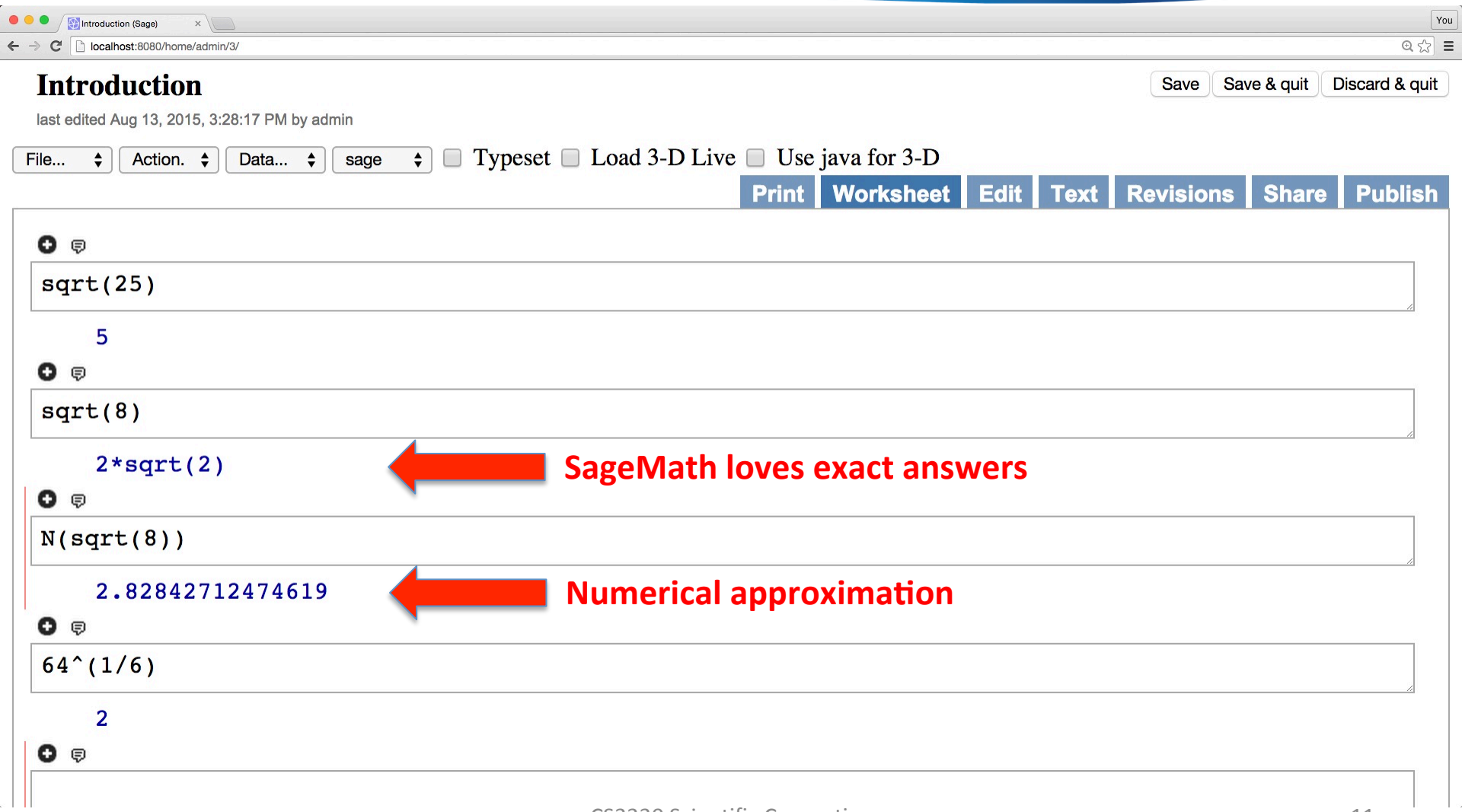


- Command-line remembers states

Use SageMath as a Calculator



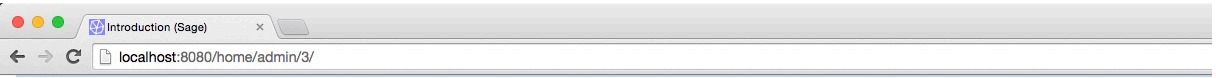
Square Root Function



Compared to Matlab

- Everything is double precision floating point by default
- Through Symbolic toolbox, which is a trimmed down version of Maple: http://www.maplesoft.com/products/Maple/

Special Constants



$$1.2345 = \underbrace{12345}_{\text{mantissa}} \times 10^{\overbrace{-4}^{\text{exponent}}}$$

Introduction

last edited Aug 13, 2015, 3:28:17 PM by admin

File... Action. Data... sage Typeset Load 3-D Live Use java for 3-D

Print Worksheet Edit Text Revisions Share Publish

```
pi*2
```

```
2*pi
```

Simplest exact value

```
e^2
```

```
e^2
```

```
e^2.5
```

```
12.1824939607035
```

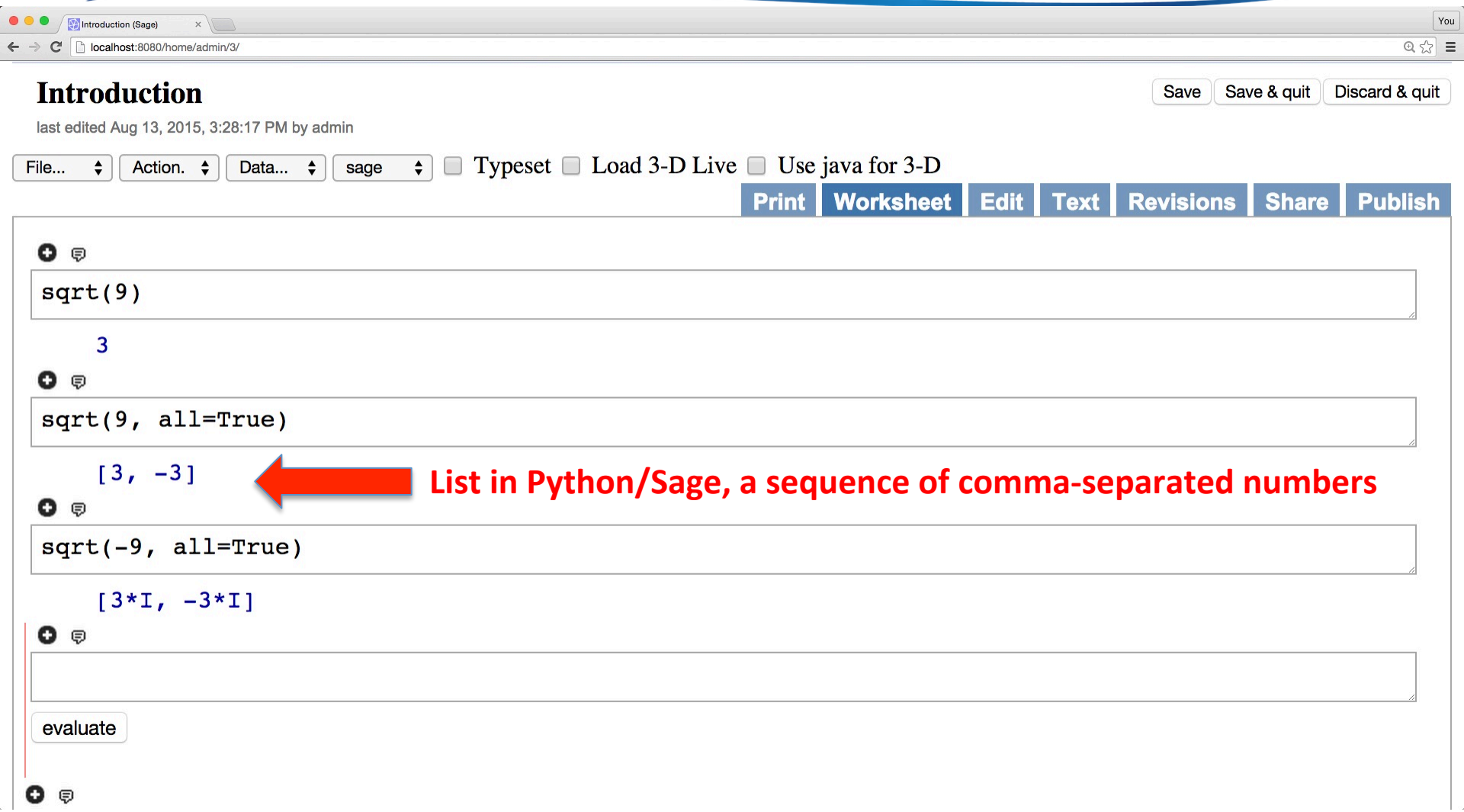
Decimals automatically lead to numerical approximation

```
N(pi, prec=256)
```

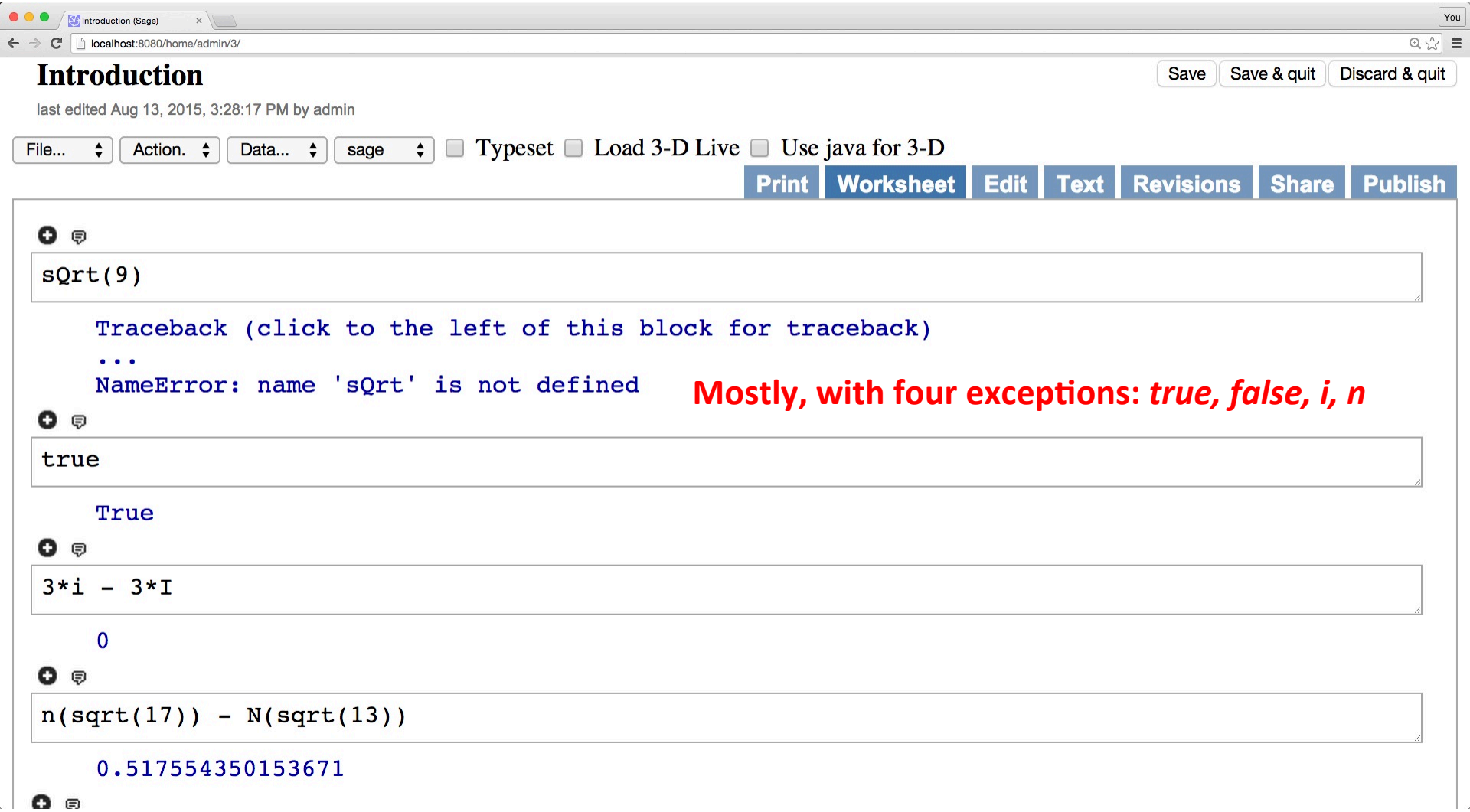
```
3.14159265358979323846206286
```

prec: (integer) default = 53: prec is the number of bits used to represent the mantissa of a floating-point number.

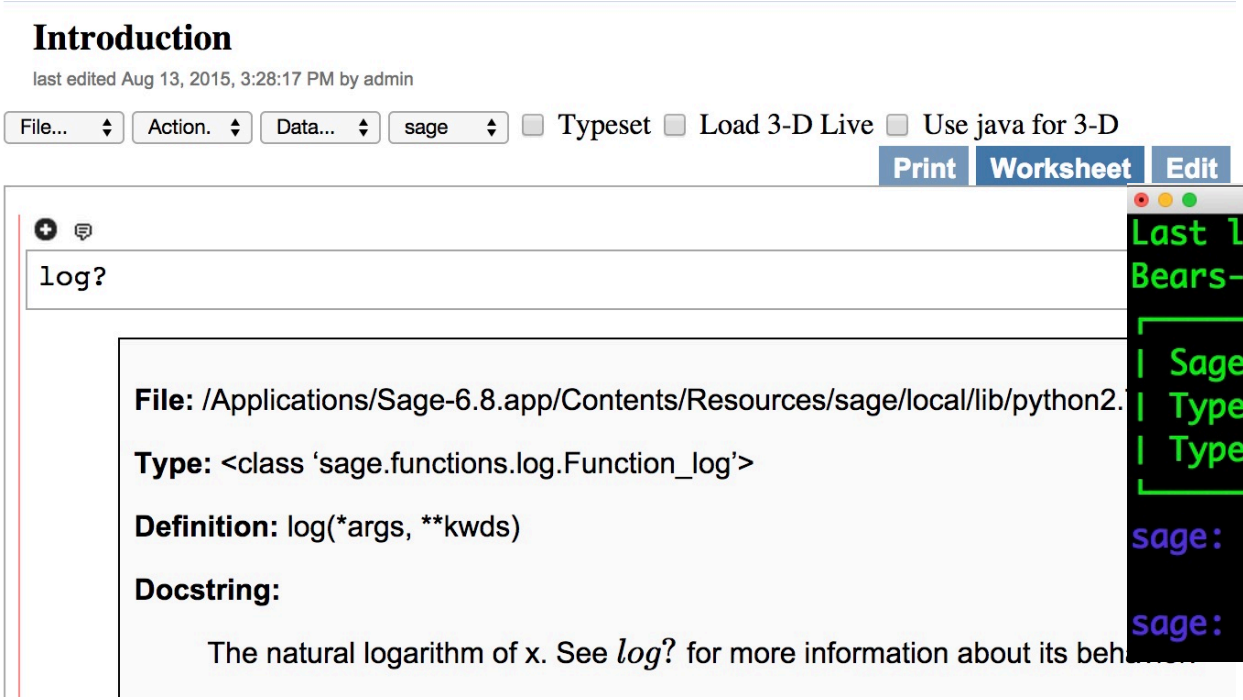
More on Square Roots

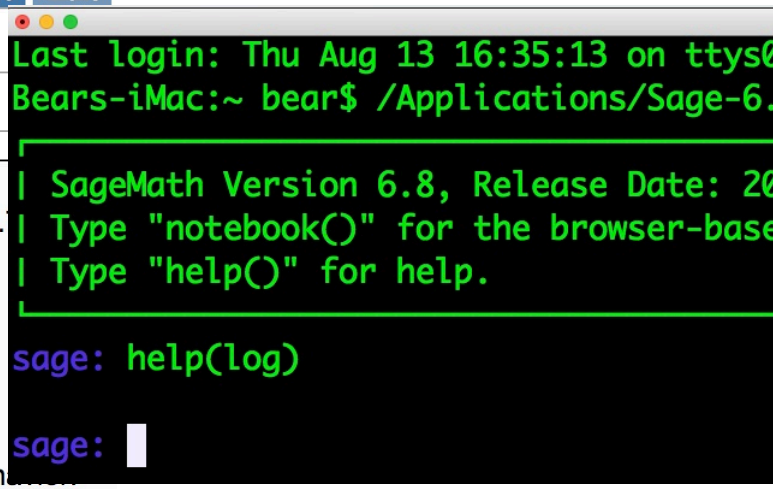


SageMath is Case Sensitive



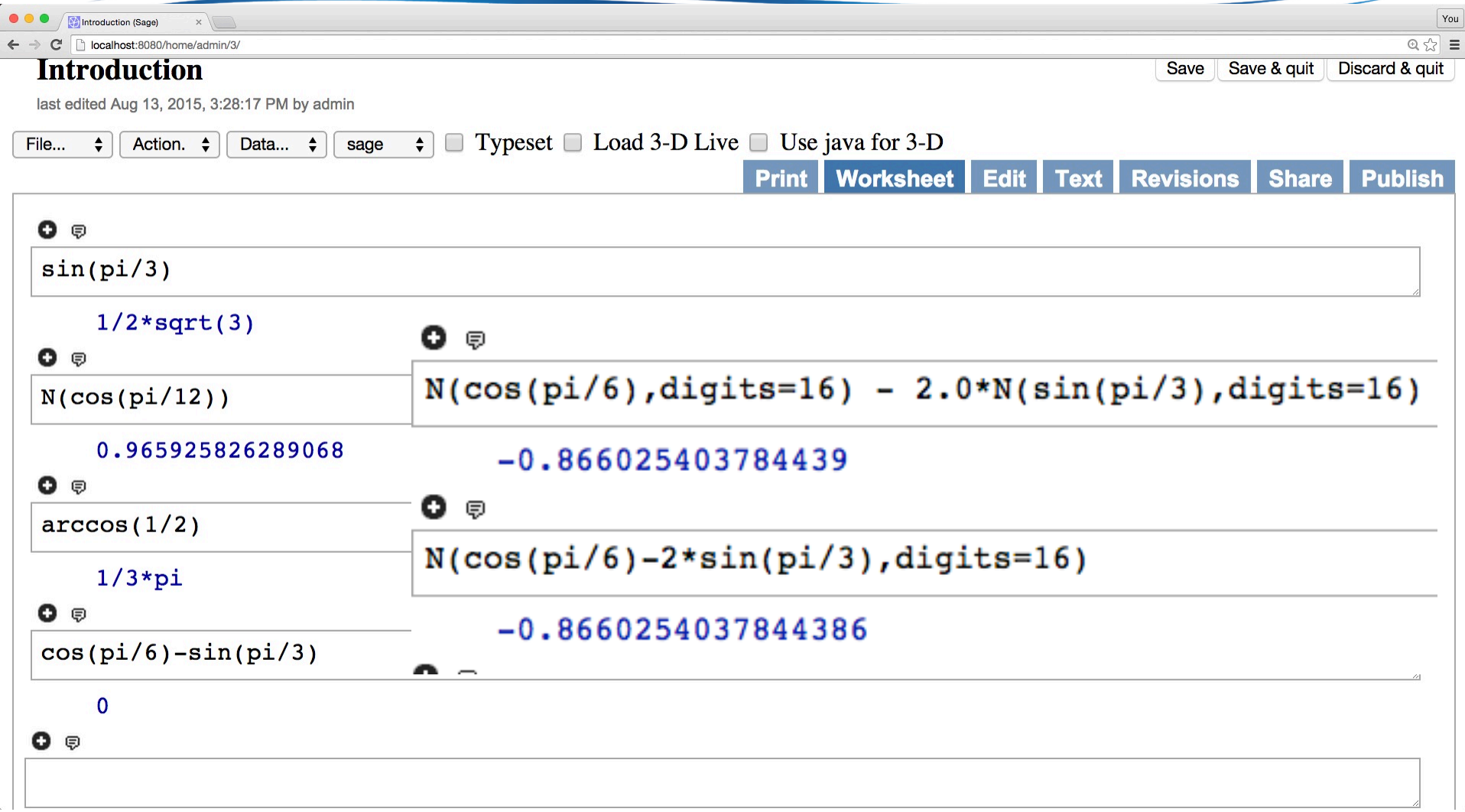
How to Get Help





- Notebook: *command*?
- Command-line: help(*command*)
- Also, utilize the *Tab-completion*

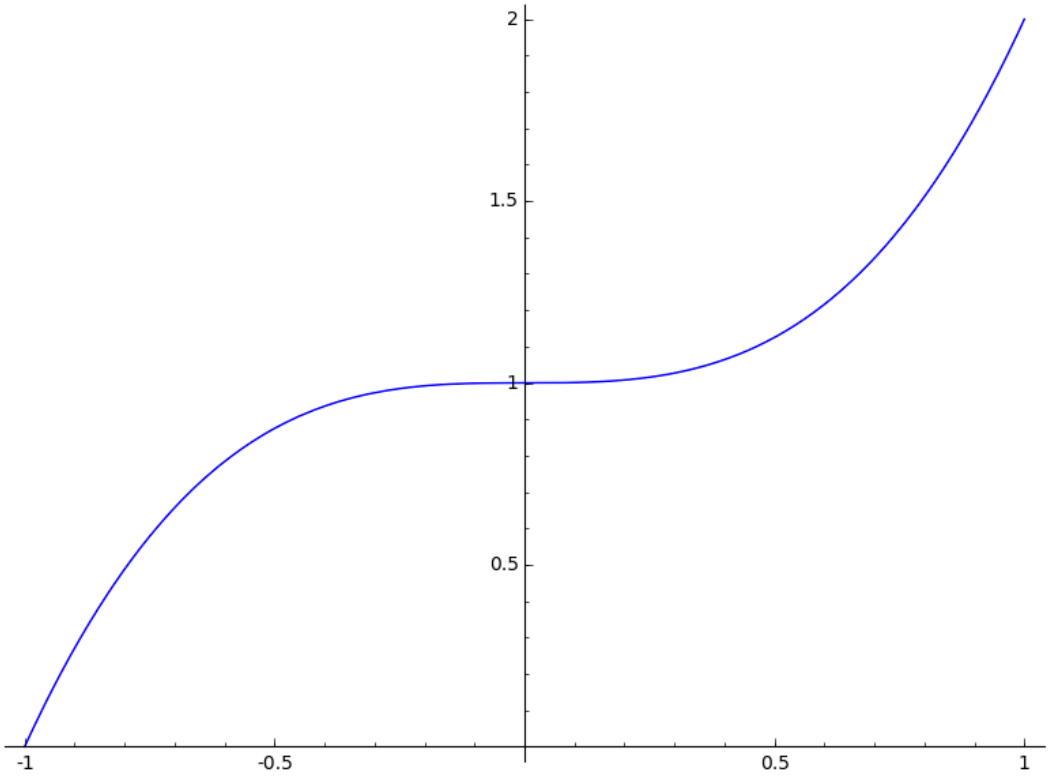
Trigonometry



Simple 2D Plots

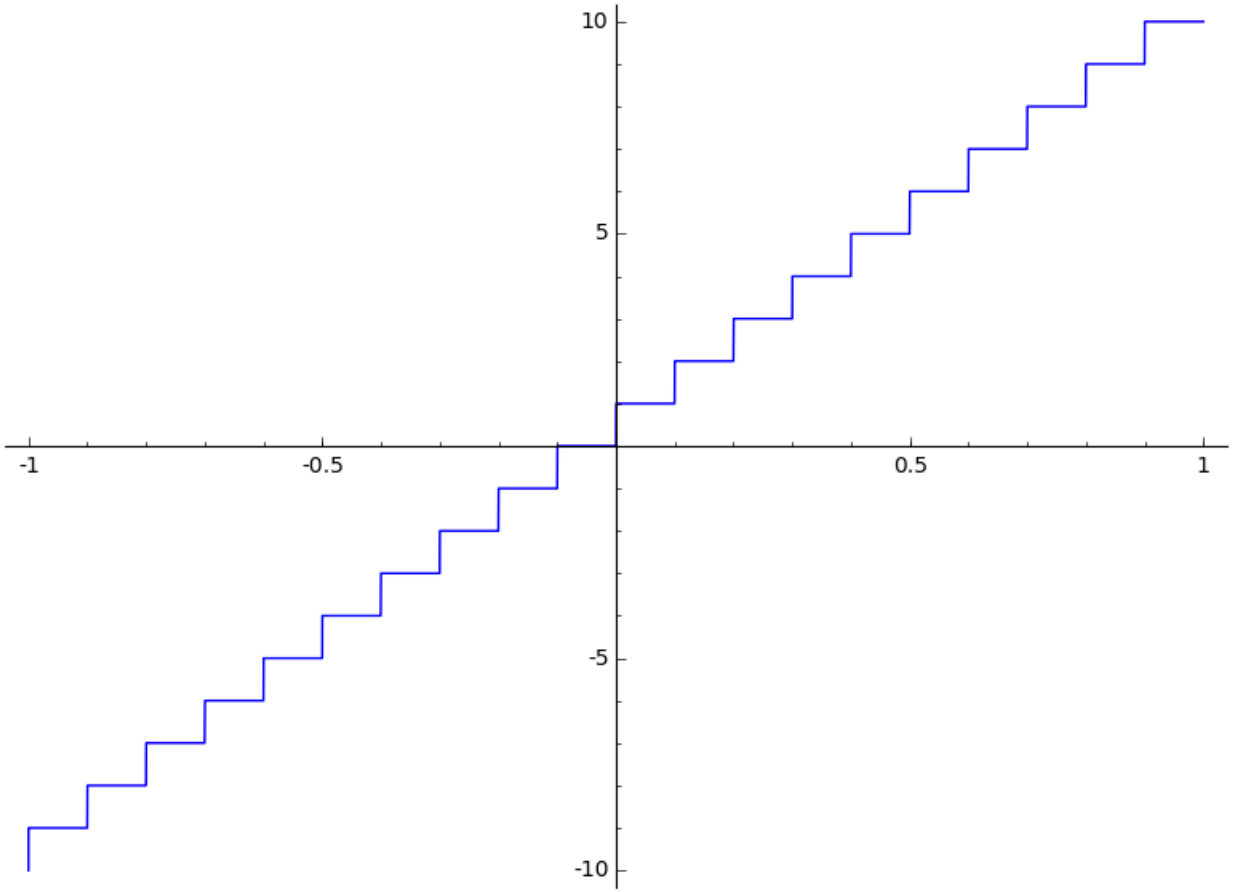
```
plot(x^3+1)
```

evaluate



More Complicated Function

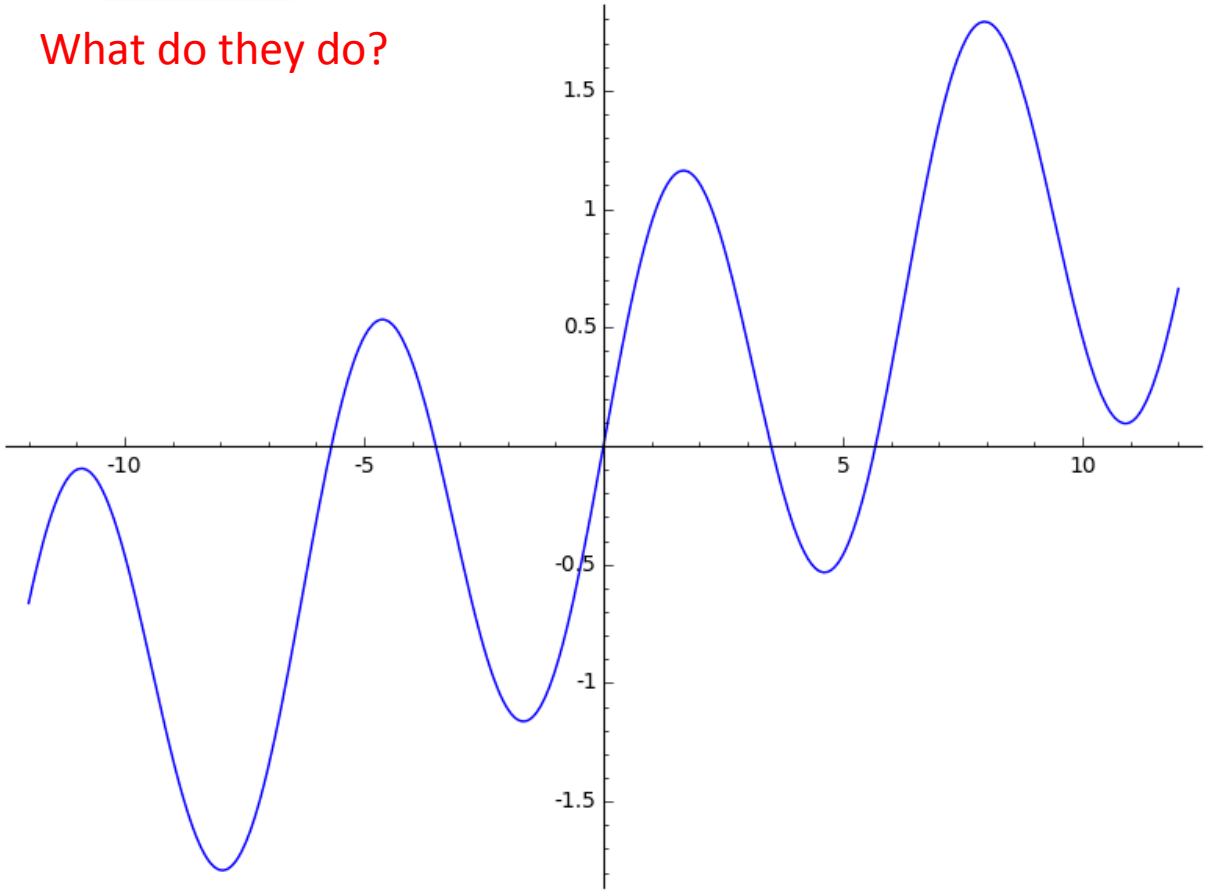
```
plot(ceil(x*10))
```



Even More Complicated

```
plot(sin(x)+0.1*x, -12, 12)
```

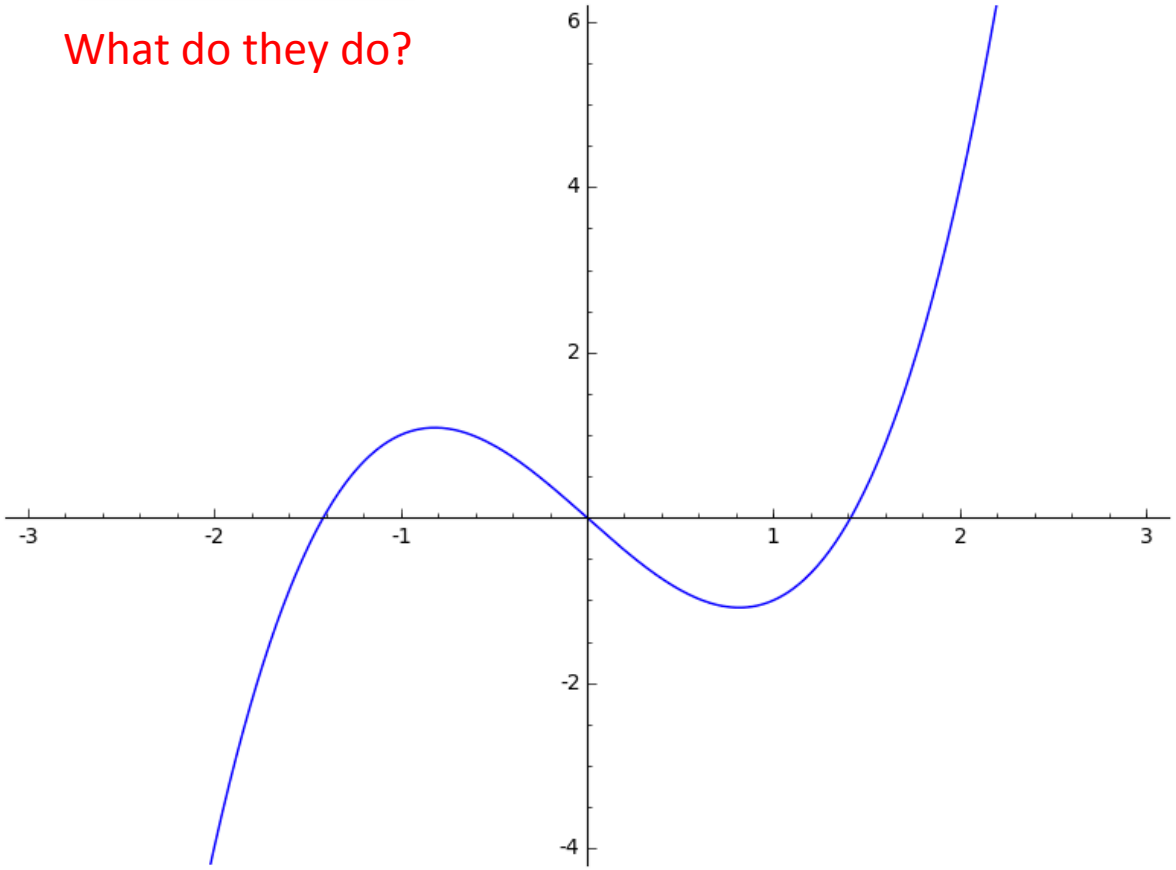
What do they do?



Ranges of Axes

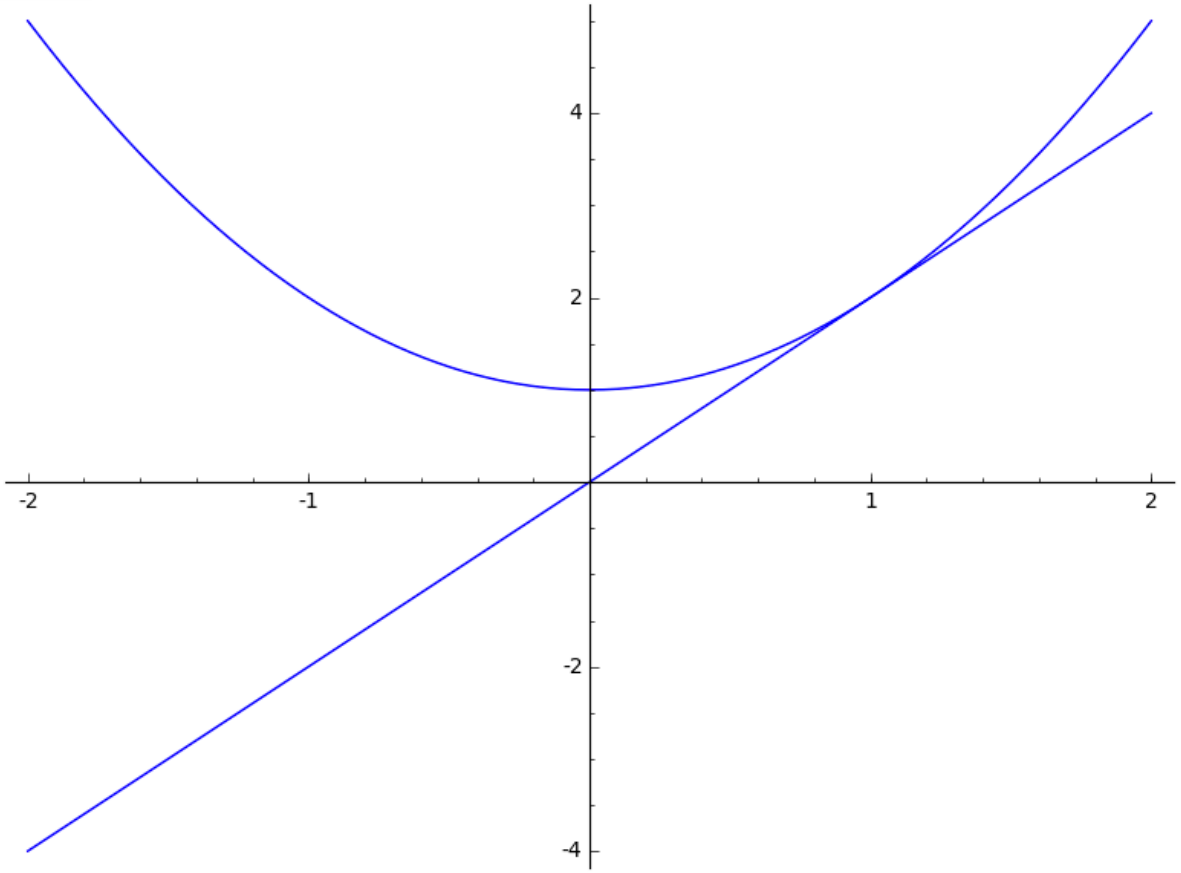
```
plot(x^3-2*x, -3, 3, ymin=-4, ymax=6
```

What do they do?



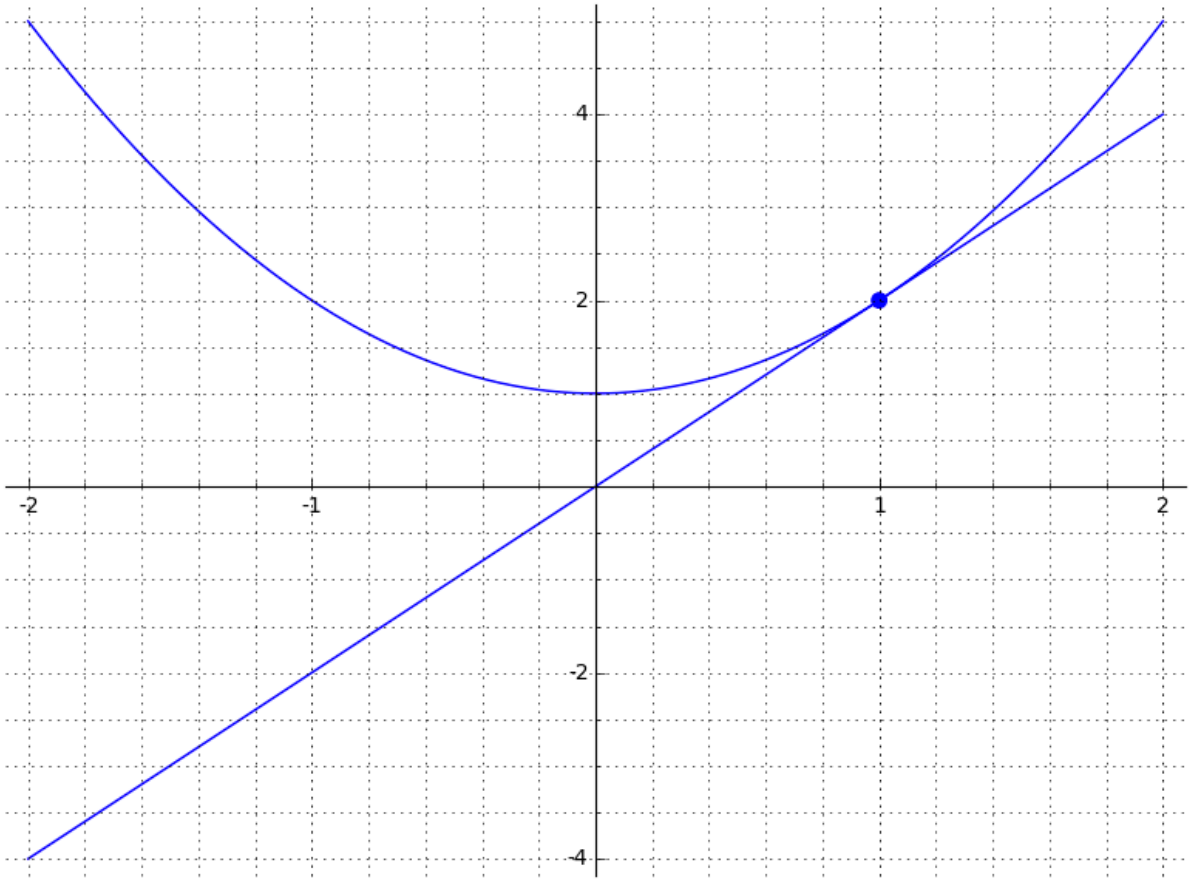
Superimposing of Multiple Graph

```
plot(2*x, -2, 2) + plot (x^2+1, -2,2)
```



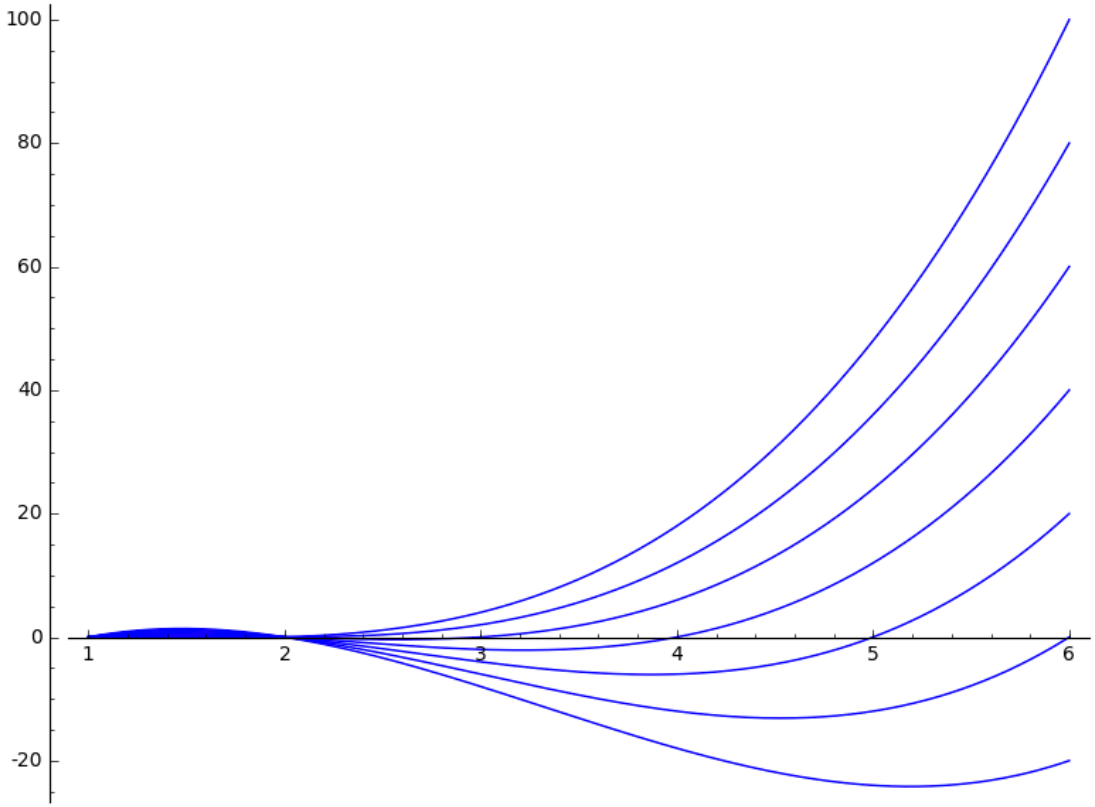
Polished Graph

```
plot(2*x, -2, 2, gridlines='minor') + plot (x^2+1, -2,2) + point ((1,2), size=60)
```



Last Example on Graphs

```
plot((x-1)*(x-2)*(x-7), 1,6)+plot((x-1)*(x-2)*(x-6), 1,6)+plot((x-1)*(x-2)*(x-5),
1,6)+plot((x-1)*(x-2)*(x-4), 1,6)+plot((x-1)*(x-2)*(x-3), 1,6)+plot((x-1)*(x-2)*(x-2),
1,6)+plot((x-1)*(x-2)*(x-1), 1,6)
```



Reduced Row Echelon Form (RREF)

- System of equations:

$$\begin{aligned} 3x - 4y + 5z &= 14 \\ x + y - 8z &= -5 \\ 2x + y + z &= 7 \end{aligned}$$

$$A = \left[\begin{array}{ccc|c} 3 & -4 & 5 & 14 \\ 1 & 1 & -8 & -5 \\ 2 & 1 & 1 & 7 \end{array}\right]$$

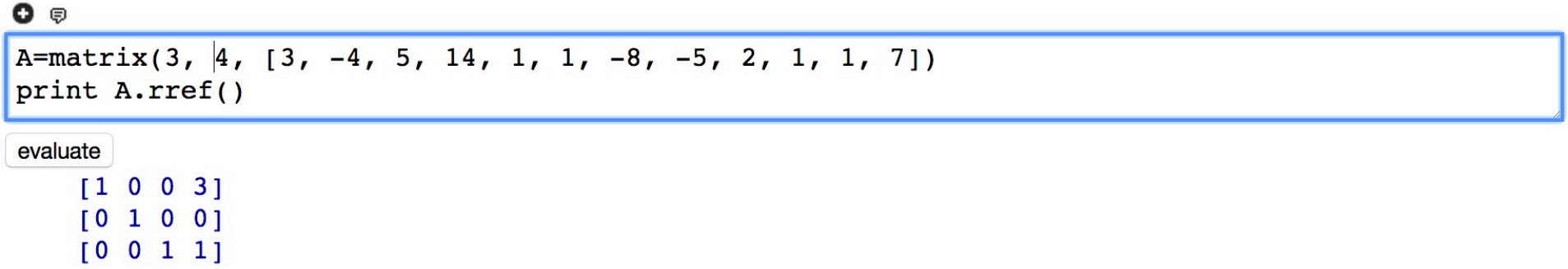
- RREF:

$$A' = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$x = 3, y = 0, z = 1$$

Getting RREF in SageMath

- Create a matrix ← what are the first two arguments of *matrix(.)*?
- Invoke *rref(.)* function



Practice, a Tricky Question

- Try to solve the following equation system using SageMath

$$2x - 5z + y = 6 + w$$

$$5 + z - y = 0$$

$$w + 3(x + y) = z$$

$$1 + 2x - y = w - 3x$$

- Answer:

$$w = \frac{-107}{7}, x = \frac{-12}{7}, y = \frac{54}{7}, z = \frac{19}{7}$$

Initializing Matrices

- Two ways to create matrices

```
B = matrix(2, 3, [1, 2, 3, 4, 5, 6])
print B
```

```
[1 2 3]
[4 5 6]
```

```
B = matrix([ [1, 2, 3], [4, 5, 6] ])
print B
```

```
[1 2 3]
[4 5 6]
```

Exceptional RREF

$$\begin{aligned} x + 2y + 3z &= 7 \\ 4x + 5y + 6z &= 16 \\ 7x + 8y + 9z &= 24 \end{aligned}$$

$$C1 = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 4 & 5 & 6 & 16 \\ 7 & 8 & 9 & 24 \end{array}\right]$$

```
B = matrix(3, 4, [1, 2, 3, 7, 4, 5, 6, 16, 7, 8, 9, 24])
print B.rref()
```

```
[ 1  0 -1  0]
[ 0  1  2  0]
[ 0  0  0  1]
```

- What is the solution?

Exceptional RREF (cont.)

$$\begin{aligned} x + 2y + 3z &= 7 \\ 4x + 5y + 6z &= 16 \\ 7x + 8y + 9z &= 25 \end{aligned}$$

$$C2 = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 4 & 5 & 6 & 16 \\ 7 & 8 & 9 & 25 \end{array}\right]$$

```
B = matrix(3, 4, [1, 2, 3, 7, 4, 5, 6, 16, 7, 8, 9, 25])
print B.rref()
```

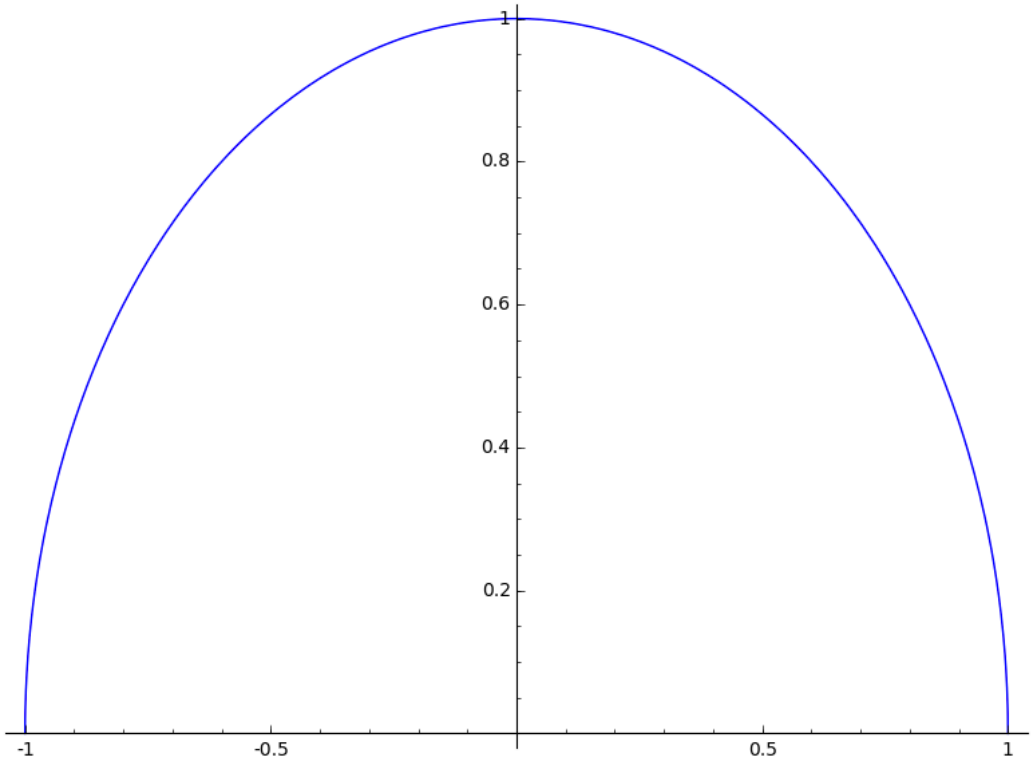
```
[ 1  0 -1 -1]
[ 0  1  2  4]
[ 0  0  0  0]
```

- What are the solutions?

Define our Own Functions

```
f(x)=sqrt(1-x^2)
print(f(0.1))
print(f(0.001))
print(f(0.00001))
print(f(0.0000001))
print(f(0.000000001))
plot(f, -1, 1)
```

```
0.994987437106620
0.999999499999875
0.999999999950000
0.999999999999995
1.00000000000000
```



Function Composition

```
f(x) = 3*x+5
g(x) = x^3 + 1
h(x) = f(g(x))
print h
h(-2)
```

```
x |--> 3*x^3 + 8
-16
```

```
f(x) = 3*x+5
g(x) = x^3 + 1
h(x) = f(g(x))
print h
h2(x) = g(f(x))
print h2
```

```
x |--> 3*x^3 + 8
x |--> (3*x + 5)^3 + 1
```

Factorization

```
print factor(12345)
print factor(987654321)
print factor(x^2 - 8*x + 15)
print gcd(x^2 - 5*x +6, x^2 - 8*x + 15)
```

```
3 * 5 * 823
3^2 * 17^2 * 379721
(x - 3)*(x - 5)
x - 3
```

Expanding a Function

```
a(x) = x^2 - 5 *x + 6
b(x) = x^2 - 8 *x + 15
f(x) = a(b(x))
print f(x)
print f.expand()
print f.factor()
```

```
(x^2 - 8*x + 15)^2 - 5*x^2 + 40*x - 69
x |--> x^4 - 16*x^3 + 89*x^2 - 200*x + 156
(x^2 - 8*x + 13)*(x - 2)*(x - 6)
```

Use Sage to Solve Problem Symbolically

- Solving $\frac{x^2}{2} - x - 2 = 0$
  - Symbolically, we get

We will start from here

$$1 \pm \sqrt{5}$$

  - Numerically, we get

$$3.23606797749979 \text{ and } -1.23606797749979$$

Single-Variable Formulas

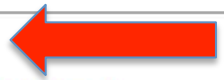
```
solve(x^2 + 3*x + 2, x)
```

```
[x == -2, x == -1]
```

```
solve(x^2 + 2*x + 3, x)
```

```
[x == -I*sqrt(2) - 1, x == I*sqrt(2) - 1]
```

```
var('alpha')
solve(cos(alpha) == 1/2, alpha)
```



Declare variable, notice that x is always a variable without explicit declaration

```
[alpha == 1/3*pi]
```

Multiple-Variable Formulas

```
var('a b c')
solve(a*x^2 + b*x + c == 0, x)
```

```
[x == -1/2*(b + sqrt(b^2 - 4*a*c))/a, x == -1/2*(b - sqrt(b^2 - 4*a*c))/a]
```

```
var('b')
solve([x+b ==6, x-b == 4], x, b)
```

```
[[x == 5, b == 1]]
```

```
var('a, b, c')
solve([9*a + 3*b +c == 32, 4*a + 2*b +c == 15, a + b + c ==6], a, b, c)
```

```
[[a == 4, b == -3, c == 5]]
```

What does it do? Remember how to solve the same problem using RREF?

Non-Linear Equation Systems

```
solve ((x-1) * (x-2) * (x-3) * (x-4) * (x-5) == 0, x)
```

```
[x == 1, x == 2, x == 3, x == 4, x == 5]
```

```
answers = solve ((x-1) * (x-2) * (x-3) * (x-4) * (x-5) == 0, x)
print answers[0]
print answers[3]
```

```
x == 1
x == 4
```

```
var('p q y')
eq1 = p+q == 9
eq2 = q*y + p*x == -6
eq3 = q*y^2 + p*x^2 == 24
eq4 = p == 2
solve([eq1, eq2, eq3, eq4], p, q, x, y)
```

List of lists? How to make it easier to read?

evaluate

```
[[p == 2, q == 7, x == -1/3*sqrt(70) - 2/3, y ==
2/21*sqrt(7)*sqrt(5)*sqrt(2) - 2/3], [p == 2, q == 7, x == 1/3*sqrt(70)
- 2/3, y == -2/21*sqrt(7)*sqrt(5)*sqrt(2) - 2/3]]
```

Higher Order Equations

```
answers = solve([x^2 - y^2 ==1, (x^2)/4 + (y^2)/3 == 1], x, y)
print answers[0]
print answers[1]
print answers[2]
print answers[3]
```

```
[x == -4/7*sqrt(7), y == -3/7*sqrt(7)]
[x == -4/7*sqrt(7), y == 3/7*sqrt(7)]
[x == 4/7*sqrt(7), y == -3/7*sqrt(7)]
[x == 4/7*sqrt(7), y == 3/7*sqrt(7)]
```

```
solve(log( x^2 ) == 5/3, x)
```

evaluate

BTW, general polynomial of degree 5 or higher have "no closed-form formula"

```
solve(sin(x+y)==0.5, x)
```

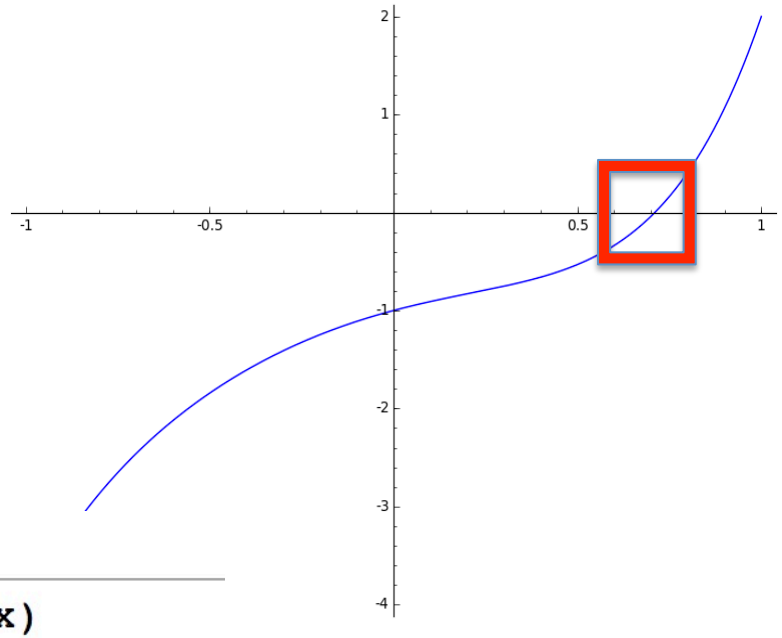
Try this to see what we get

```
[x == 1/6*pi - y]
```

```
var('a0, a1, a2, a3, a4')
solve(a4*x^4 + a3*x^3 + a2*x^2 + a1*x + a0 == 0, x)
```

Quantic Polynomials and Numerical Solver

Solve $x^5 + x^4 + x^3 - x^2 + x - 1 = 0$




```
solve(x^5 + x^4 + x^3 - x^2 + x - 1 == 0, x)
```

```
[0 == x^5 + x^4 + x^3 - x^2 + x - 1]
```

Why can't we find a solution?

```
find_root(x^5 + x^4 + x^3 - x^2 + x - 1, -100, 100)
```

```
0.7104342557869131
```

More Numerical Solver Examples

```
find_root(e^x - 1/x, -1, 1)
```

```
0.5671432904097954
```

```
find_root(cos(x)==sin(x), 0, pi/2)
```

evaluate

```
0.7853981633974484
```

```
var('n')
find_root(7000 == 5000*(1+0.045/12)^n, 0, 100000)
```

```
89.89406093308004
```

Exercise: what are we calculating?

How to Take Derivatives

```
diff(x^4 - x^2 + x , x)
```

```
4*x^3 - 2*x + 1
```

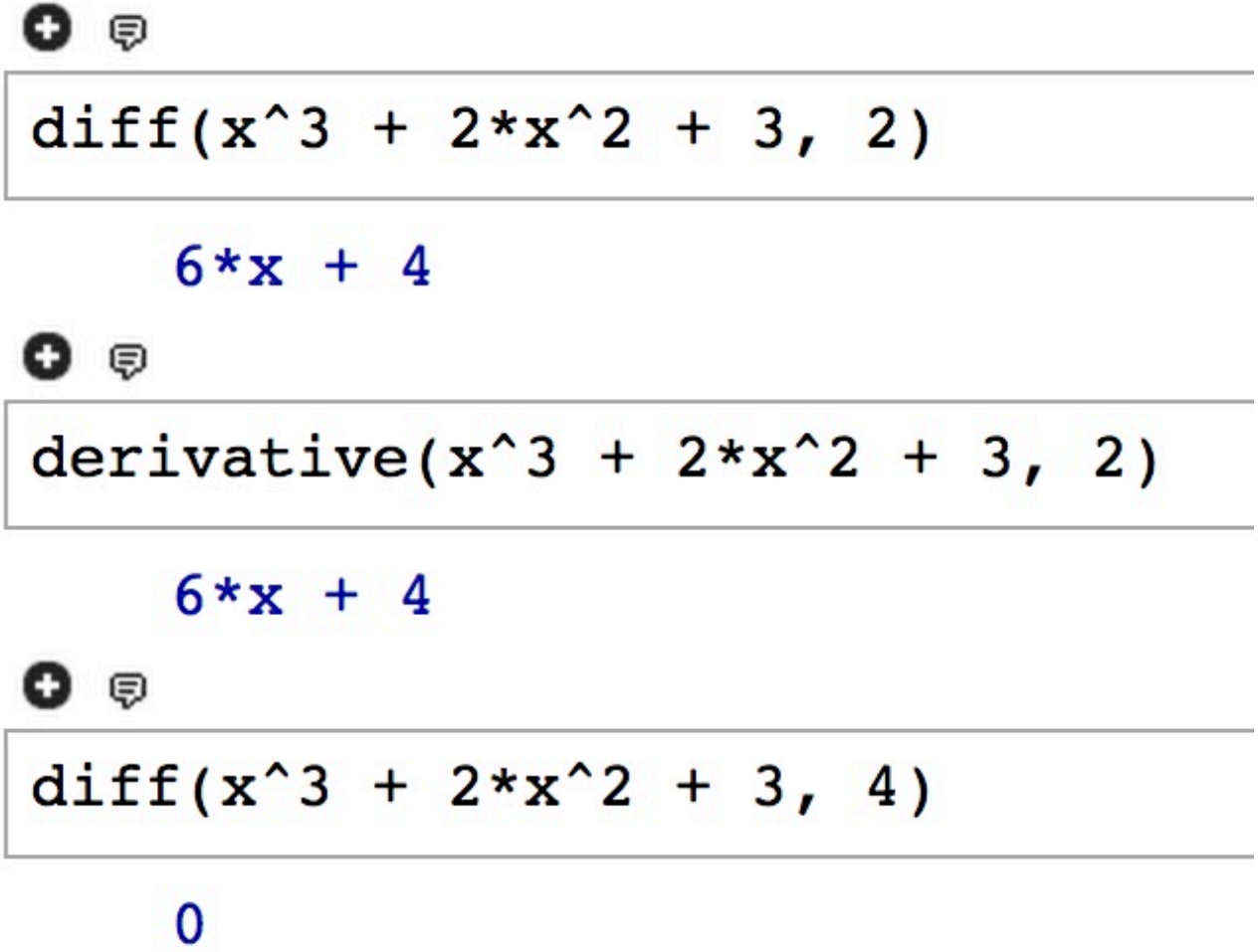
```
diff(sin(x/e), x)
```

```
cos(x*e^(-1))*e^(-1)
```

```
g(x)=e^(-10*x)
g.derivative()
f(x) = g.derivative()
f(-12)
```

```
-10*e^120
```

Higher Order Derivatives



Indefinite Integral

```
integral(x*cos(x^2), x)
```

```
1/2*sin(x^2)
```

$$\int x\cos(x^2)dx = \frac{1}{2}sin(x^2)$$?

```
integral(x / (x^2 + 2), x)
```

```
1/2*log(x^2 + 2)
```

$$\int \frac{x}{x^2+2}dx = \frac{1}{2}\log(x^2+2)$$?

Don’t forget the constant!

Definite and Impossible Integrals

```
integral(x*cos(x^2), x, -2, 3)
```

```
1/2*sin(9) - 1/2*sin(4)
```

$$\int_{-2}^{3} x\cos(x^2)dx$$

```
integral(x / (x^2 + 2), x, 1, 2)
```

```
1/2*log(6) - 1/2*log(3)
```

$$\int_{1}^{2} \frac{x}{x^2+2}dx$$

```
var('y')
assume(y>0)
integral(2 / sqrt(pi) * (e ^ (-x^2)), x, 0, y)
```

$$\int_{0}^{y} \frac{2}{\sqrt{\pi}}e^{-x^2}$$

We even define a special function for this!

Numerical Integrations

```
integral(exp(-x^3) * sin(x^2), x, 1, 3)
```

```
integrate(e^(-x^3)*sin(x^2), x, 1, 3)
```

```
numerical_integral(x / (x^2 + 2), 1, 2)
```

```
(0.3465735902799726, 3.847739796558e-15)
```

Summary

- We introduced SageMath, an opensource project based on Python
- We go over basics of SageMath, including symbolic and numerical solutions, matrices, and simple plots
- We will mostly use SageMath for symbolic solutions
  - SageMath was initially designed for this
  - Other tasks are done in Matlab/Octave
- References:
  - http://www.sagemath.org ← Official Web and resources
  - http://www.gregorybard.com/SAGE.html ← Our textbook

SageMath #1 Homework (S1)

1. (1%) Find at least 5 real number solutions of $e^{-x^2}\sin(16x) = 0$ in *[-1, 1]* using SageMath
   - Hint: Plot the curve first!
2. (1%) Run *A = random_matrix(ZZ, 3, 4)* in SageMath. Map the matrix into an equation system (say variables are *x*, *y*, and z*)*. Write down the equation system (in Latex, please turn in the .tex and .pdf). Solve the equation system using SageMath. Present the solution in terms of x, y, and z.
3. (1%) Execute the following code in SageMath. Explain why the two prints give different answers? You need to identify the key difference to get the point.

```
x = ceil(random() * 888)
print sqrt(x)
print N(sqrt(x))
```

Preview of Midterm #1

midterm1.pdf

Create | 1 / 2 | 195% | Tools | Comment | Share

Midterm #1 (10%)

CS3330 Scientific Computing, Instructor: Cheng-Hsin Hsu

Department of Computing Science, National Tsing Hua University, Taiwan

1:20 p.m. – 3:10 p.m., Oct. 21th, 2016

- **Please create a new latex document, write your solution (no need to copy the questions, but please clearly mark the question numbers in order) into it, typeset it, and submit both your .tex and .pdf files before you leave the classroom. No partial credits will be given to students who fail to submit his/her .pdf file.**
- **You are allowed (actually encouraged) to search online for tips.**
- **You are not allowed to copy and paste source codes from the Internet. Furthermore, you cannot exchange (online/offline) messages with any of your peers during the exam. These are considered as academic dishonesty, which automatically leads to zero point. Furthermore, we will have no choice but report this incident to the university.**

1) (2%) Please reproduce the following formula in Latex