Sample Solutions of HW of Chapter 2: Systems of Linear Equations

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Note that, the solutions are for your reference only. If you have any doubts about the correctness of the answers, please let the instructor and the TA know. More importantly, like other math questions, the homework questions may be solved in various ways. Do not assume the sample solutions here are the only *correct* answers; discuss with others about alternate solutions.

We will not grade your homework assignment, but you are highly encouraged to discuss with us during the Lab hours. The correlation between the homework assignments and quiz/midterm/final questions is high. So you do want to practice more and sooner.

1 Review Questions

- **2.5** False. For example, let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, then $det(A) = -1 \neq 0$. Thus, A is not singular.
- **2.10** True. row dependent $\Rightarrow det(A) = 0 \Rightarrow$ singular $\Rightarrow$ column dependent
- **2.15** False. See example 2.14, $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, and A is singular since $det(A) = 0$. However, $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = LU$
- **2.20** True. The complexity of explicit matrix inversion is about $O(n^3)$, and the majority of the work is due to factorization.
- **2.25** True. $cond(A) = \|A\| \cdot \|A^{-1}\| = \|A^{-1}\| \cdot \|A\| = cond(A^{-1})$
- **2.30** They are the same (bad), as A is singular in either case. The back-substitution will fail because of that.
- **2.35** Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0.5 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix} \Rightarrow A \begin{bmatrix} 3 \\ 2 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0.5 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \end{bmatrix}$

- **2.40** (1) When there is a zero on the diagonal, the multiplier m is undefined. (2) Even without zero element on the primary diagonal, the multiplier may be very (arbitrarily) large, which leads to overflow in subsequent calculations. That's why we need pivoting.

- **2.45** Firstly, we should compute Bc and get the result, say d. Then we compute $A^{-1}d$ by LU factorization of A, followed by n forward- and back-substitutions.

- **2.50** Yes, Let $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then $\|x\|_1 = 1 + 0 = 1 = \|x\|_\infty = max\{1, 0\}$

- **2.55** Computing the condition number of a general matrix is nontrival because the task involves the computing of the inverse of the matrix.

- **2.60** $fl(A)$ is (not singular, but) nearly singular, so that condition number shall be very large.

- **2.65** $cond(A) \geq 10^{12}$

- **2.70**
(a) $2n - 1$: n to store the entries of the first row, $n - 1$ to store the scalar γ of the i row such that $\gamma row_1 = row_i$, where $i \neq 1$.
(b) $3n - 2$: include n multiplications and $n - 1$ additions when computing the first row and $n - 1$ multiplications when letting the scalar multiply the answer of the first row.

- **2.75** n square roots are required to compute the Cholesky factorization of an $n \times n$ symmetric positive definite matrix.

- **2.80**
(a) $O(n^2)$
(b) $O(n^2)$
(c) $O(n^3)$ i.e., we have to compute the whole linear system again.

2 Exercises

- **2.3** Since A is singular, the solution of $Ax = b$ is not unique. Thus, there are two nonzero solutions x and y, $x \neq y$, such that $Ax = Ay = b$. Let $z = x - y$, then $Az = A(x - y) = Ax - Ay = b - b = o$. Moreover, for any scalar γ, we have a solution $x + \gamma z$ such that $A(x + \gamma z) = Ax + A\gamma z = Ax + \gamma o = b$.

- **2.7**
(a) $det(A) = 1 - (1 + \epsilon)(1 - \epsilon) = 1 - (1 - \epsilon + \epsilon - \epsilon^2) = \epsilon^2$
(b) In floating-point arithmetic, when $|\epsilon| < \sqrt{\epsilon_{mach}}$, the computed value of $det(A)$ is zero.
(c) $A = \begin{bmatrix} 1 & 1+\epsilon \\ 1-\epsilon & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1+\epsilon & 1 \end{bmatrix} \begin{bmatrix} 1 & 1+\epsilon \\ 0 & \epsilon^2 \end{bmatrix} = LU$

(d) In floating-point arithmetic, when $|\epsilon| < \sqrt{\epsilon_{mach}}$, the computed value of U is zero.

- **2.16**
  (a) $\begin{bmatrix} 1 & a \\ c & b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -c & 1 \end{bmatrix} \begin{bmatrix} 1 & a \\ 0 & b-ac \end{bmatrix} = LU$
  (b) When $b - ac = 0$, this matrix is singular (Since $det(A) = b - ac = 0$.)

- **2.27** Notice that $v^T A^{-1} u$ and $(1 - v^T A^{-1} u)^{-1}$ are scalars.
  $(A - uv^T)(A^{-1} + A^{-1}u(1 - v^T A^{-1}u)^{-1} v^T A^{-1})$
  $= I + \frac{uv^T A^{-1}}{1 - v^T A^{-1} u} - uv^T A^{-1} - \frac{uv^T A^{-1} uv^T A^{-1}}{1 - v^T A^{-1} u}$
  $= I + \frac{uv^T A^{-1}}{1 - v^T A^{-1} u} + \frac{-uv^T A^{-1} + uv^T A^{-1} v^T A^{-1} u}{1 - v^T A^{-1} u} - v^T A^{-1} u \frac{uv^T A^{-1}}{1 - v^T A^{-1} u}$
  $= I + \frac{uv^T A^{-1}}{1 - v^T A^{-1} u} - \frac{uv^T A^{-1}}{1 - v^T A^{-1} u} + v^T A^{-1} u \frac{uv^T A^{-1}}{1 - v^T A^{-1} u} - v^T A^{-1} u \frac{uv^T A^{-1}}{1 - v^T A^{-1} u}$
  $= I$

- **2.35** $A = BB^T = (BB^T)^T = A^T \Rightarrow A$ is symmetric. Let $x \neq o$ be a vector, $x^T A x = x^T B B^T x = (B^T x)^T (B^T x) \geq 0$. However, $B^T x \neq 0$ since B is nonsingular and $x \neq 0$. Thus, A is positive definite.

- **2.42** (a) Given an upper triangular matrix U, the following algorithm overwrites U with U^{-1}, accessing only the upper triangular portion of the array. An analogous algorithm works for a lower triangular matrix.

  ```
  for k = n to 1
      u_kk = 1/u_kk
      for i = k − 1 to 1
          t = 0
          for j = i + 1 to k
              t = t + u_ij u_jk
          end
          u_ik = −t/u_ii
      end
  end
  ```

  (b) Given a matrix A, one could compute its LU factorization $A = LU$ in place, overwriting the upper triangle of A with U and the strict lower triangle of A with the strict lower triangle of L, not storing the unit diagonal of L. The triangular matrices U and L could then be inverted in place using the algorithms from part(a). This would effectively give one a *representation* of A^{-1}, but to obtain A^{-1} explicitly, one would have to compute $A^{-1} = U^{-1} L^{-1}$ in place, and this is impossible because there is no order in which to compute the product that does not overwrite some entries that will still be needed subsequently.