Sample Solutions of HW of Chapter 5: Nonlinear Equations

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Note that, the solutions are for your reference only. If you have any doubts about the correctness of the answers, please let the instructor and the TA know. More importantly, like other math questions, the homework questions may be solved in various ways. Do not assume that the sample solutions here are the only *correct* answers; discuss with others about alternate solutions.

We will not grade your homework assignment, but you are highly encouraged to discuss with us during the Lab hours. The correlation between the homework assignments and quiz/midterm/final questions is high. So you do want to practice more and sooner.

1 Review Questions

- **5.5** It is better to terminate the iteration when $|x_k - x_{k-1}|$ is small. The reason is that for an ill-conditioned problem, $|f(x_k)|$ can be small without x_k being close to the true solution.

- **5.10** Since f is continuous in $[a, b]$, by *Intermediate Value Theorem*, if $f(a)$ and $f(b)$ differ in their signs then there is a root x^* in $[a, b]$ such that $f(x^*) = 0$.

- **5.15**

  (a) A smooth function f if $f(x^*) = f'(x^*) = \cdots = f^{(m-1)}(x^*) = 0$, but $f^{(m)} \neq 0$, where $m > 1$, then x^* is a multiple root with multiplicity m.

  (b) The convergence rate is at least quadratic.

  (c) The convergence rate is linear, with constant $C = 1 - (1/m)$, where m is the multiplicity.

- **5.20**

  (a) If $x^* = g(x^*)$ and $|g'(x^*)| < 1$, then the iteration scheme $x_{k+1} = g(x_k)$ is locally convergent to x^*.

  (b) The convergent rate is linear, with constant $C = |g'(x^*)|$.

  (c) $g'(x^*) = 0$

  (d) Yes, $g(x) = x - f(x)/f'(x)$, where $f'(x) \neq 0$.

- **5.25**

(a) 1 The polynomial may not have any real roots.
2 Even if it has real roots, the roots may not be easy to compute.
3 Moreover, it may not be easy to choose which root to use in the next iteration.

(b) An alternative approach to get around these difficulties is provided by *inverse interpolation*.

- **5.30** *Inverse interpolation*, in which one fits the values x_k as a function of the values $y_k = f(x_k)$ by a polynomial $p(y)$. It is useful, because the next approximation solution is simply $p(0)$.

- **5.35**

Method	Advantage	Disadvantage
Form companion matrix of polynomial and use eigenvalue routine to compute all its eigenvalues	Reliable	Quite expensive in terms of storage and computations.

Other simpler methods include: solving the polynomial for x_i, and divide the original polynomial $f(x)$ with $(x - x_i)$, and solve the new polynomial. Repeat it until we get enough number of roots.

2 Exercises

- **5.2**

	$f(x)$	$f'(x)$	$x_{k+1} = x_k - f(x_k)/f'(x_k)$
(a)	$x^3 - 2x - 5$	$3x^2 - 2$	$x_{k+1} = x_k - (x_k^3 - 2x_k - 5)/(3x_k^2 - 2)$
(b)	$e^{-x} - x$	$-e^{-x} - 1$	$x_{k+1} = x_k + (e^{-x_k} - x_k)/(e^{-x_k} + 1)$
(c)	$x\sin(x) - 1$	$x\cos(x) \ + sin(x)$	$x_{k+1} = x_k - (x_k\sin(x_k) - 1)/(x_k\cos(x_k) \ + \sin(x_k))$

- **5.5**

(a)

$$\begin{aligned} x_{k+1} &= x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})} \\ &= \frac{x_k(f(x_k) - f(x_{k-1})) - f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})} \\ &= \frac{x_k f(x_k) - x_k f(x_{k-1}) - f(x_k)x_k + f(x_k)x_{k-1}}{f(x_k) - f(x_{k-1})} \\ &= \frac{f(x_k)x_{k-1} - x_k f(x_{k-1})}{f(x_k) - f(x_{k-1})} = Right \end{aligned}$$

(b) The formula in part (a) gives the new iteration as a quotient of two differences, each is between two quantities that are nearly equal, and hence may suffer substantial cancellation. The standard formula gives the new iteration as a small perturbation to the previous iteration, and only the perturbation computation suffers from potential cancellation.

- **5.7**

We seek an x such that $f(x) = \Gamma(x) - 1.5 = 0$. Let $a = 0.5, b = 1.0, c = 0.75$. Then

$$\begin{aligned}
f_a &= f(a) = \Gamma(a) - 1.5 = \sqrt{\pi} - 1.5 \approx 0.2725 \\
f_b &= f(b) = \Gamma(b) - 1.5 = 1 - 1.5 = -0.5 \approx 0.2725 \\
f_c &= f(c) = \Gamma(c) - 1.5 = \sqrt{\pi}/2 - 1.5 \approx -0.6138
\end{aligned}$$

(a) We fit a quadratic polynomial $p(t) = \alpha + \beta t + \gamma t^2$ to $(a, f(a))$, $(b, f(b))$, and $(c, f(c))$, and use one of its roots as an approximation to x. To determine the coefficients α, β and γ of $p(t)$.
We solve the system of linear equations:

$$\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} f_a \\ f_b \\ f_c \end{bmatrix}$$

and obtain $\alpha \approx 1.7036$, $\beta \approx -3.5210$, $\gamma \approx 1.3174$. Thus, $p(t) = 0$ has roots $t \approx 0.6344$ and $t \approx 2.0383$, one of which is chosen as an approximation to x.

(b) Using the formulas in Section 5.5.5 for inverse quadratic interpolation,

$$\begin{aligned}
u &= f_b/f_c \approx 0.8146, \\
v &= f_b/f_a \approx -1.8352, \\
w &= f_a/f_c \approx -0.4439, \\
p &= v(w(u - w)(c - b) - ((1 - u)(b - a)) \approx 0.6827, \\
q &= (w - 1)(u - 1)(v - 1) \approx -0.7588, \\
x &= b + p/q \approx 0.1003,
\end{aligned}$$

(c) Using the procedure in Section 5.5.6 for linear fractional interpolation,

$$x = c + h = c + \frac{(a - c)(b - c)(f_a - f_b)f_c}{(a - c)(f_c - f_b)f_a - (b - c)(f_c - f_a)f_b} \approx c - 0.9386 \approx 0.5614.$$

Note: $\Gamma(0.59533) \approx 1.5$.

- **5.10** The first iteration of Newton's method is $x1 = x0 + s0$, where $s0$ is determined by solving the linear system $J_f(x0)s0 = -f(x0)$. For this problem, we have

$$J_f(x) = \begin{bmatrix} 2x_1 & -2x_2 \\ 2x_2 & 2x_1 \end{bmatrix} = -f(x) = \begin{bmatrix} x_1^2 - x_2^2 \\ 2x_1x_2 - 1 \end{bmatrix}$$

$$\begin{aligned} &\Rightarrow J_f(x_0) = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} = -f(x_0) = -\begin{bmatrix} -1 \\ -1 \end{bmatrix} \\ &\Rightarrow J_f(x_0)s_0 = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} s_{01} \\ s_{02} \end{bmatrix} = -\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &\Rightarrow s_0 = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix} \end{aligned}$$

Therefore, we have $x_1 = x_0 + s_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$.

- **5.15** Solving the 3x3 system of linear equations from Section 5.5.6 gives.

$$\begin{aligned} w &= \frac{(c-a)(bf_b - af_a) - (b-a)(cf_c - af_a)}{(f_a - f_c)(bf_b - af_a) - (f_a - f_b)(cf_c - af_a)} \\ v &= \frac{b-a}{bf_b - af_a} - w\frac{f_a - f_b}{bf_b - af_a} = \frac{c(f_a - f_b) + a(f_b - f_c) + b(f_c - f_a)}{af_a(f_b - f_c) + cf_c(f_a - f_b) + bf_b(f_c - f_a)} \\ u &= a - vaf_a + wf_a = \frac{bcf_a(f_c - f_b) + acf_b(f_a - f_c) + abf_c(f_b - f_a)}{af_a(f_b - f_c) + cf_c(f_a - f_b) + bf_b(f_c - f_a)} \end{aligned}$$

Then

$$\begin{aligned} h &= u - c \\ &= \frac{(a-c)(c-b)(f_a - f_b)f_c}{af_a(f_b - f_c) + cf_c(f_a - f_b) + bf_b(f_c - f_a)} \\ &= \frac{-(a-c)(b-c)(f_a - f_b)f_c}{af_a(f_b - f_c) + cf_c(f_a - f_b) + bf_b(f_c - f_a) + cf_af_b - cf_af_b} \\ &= \frac{(a-c)(b-c)(f_a - f_b)f_c}{(a-c)(f_c - f_b)f_a - (b-c)(f_c - f_a)f_b} \end{aligned}$$