Sample Solutions of HW of Chapter 3: Linear Least Squares

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Note that, the solutions are for your reference only. If you have any doubts about the correctness of the answers, please let the instructor and the TA know. More importantly, like other math questions, the homework questions may be solved in various ways. Do not assume that the sample solutions here are the only *correct* answers; discuss with others about alternate solutions.

We will not grade your homework assignment, but you are highly encouraged to discuss with us during the Lab hours. The correlation between the homework assignments and quiz/midterm/final questions is high. So you do want to practice more and sooner.

1 Exercises

- **3.1**

  (a) $Ax = \begin{bmatrix} 1 & 10 \\ 1 & 15 \\ 1 & 20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cong \begin{bmatrix} 11.60 \\ 11.85 \\ 12.25 \end{bmatrix} = b$

  (b) This system is inconsistent. Using the first two equations, $x = [11.1, 0.05]^T$. Using the first and third equations, $x = [10.95, 0.065]^T$. Using the second and third equations, $x = [10.65, 0.08]^T$. There is no rigorous reason to prefer any one of these solutions over the others, although the second solution might seem "safest" in that, it is closest to the average over all the solutions.

  (c) The system of normal equations is

  $$A^T Ax = \begin{bmatrix} 3 & 45 \\ 45 & 725 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 35.7 \\ 538.75 \end{bmatrix} = A^T b,$$

  which has solution $x = [10.925, 0.065]^T$. This answer is closest to the second solution obtained in (b), using the first and third equations.

- **3.7**

  (a) The function $\phi(y) = \|b - y\|_2$ is continuous and coercive, and therefore has a minimum on the closed, unbounded set $span(A)$. Thus, by definition of $span(A)$, there is some x that minimizes $\|b - Ax\|_2$, which is therefore a solution to the least squares problem.

(b) Suppose that x_1 and x_2 are two solutions, and let $z = x_2 - x_1$. Then, since $Ax_1 = y = Ax_2$, we have $Az = o$. Now if $z \neq o$, i.e., $x_1 \neq x_2$, then $rank(A) < n$. For the other direction, if $rank(A) < n$, then there is a nonzero z such that $Az = o$, and hence if x is a least squares solution, then $x + z$ is also a solution, since$A(x + z) = Ax = y$. We conclude that the least squares solution is unique if, and only if, $rank(A) = n$.

- **3.12**

(a)

(1) and (2) $\Rightarrow$ (3): $A^2 = A^T A = I$.

(1) and (3) $\Rightarrow$ (2): $A^T A = A^2 = I$.

(2) and (3) $\Rightarrow$ (1): $A^T = A^T A^2 = A$.

(b) $\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) Householder transformations.

- **3.23** We compute $\boldsymbol{R}$ in floating-point arithmetic using Givens rotations. To zero the $(2,1)$ entry $t = \epsilon$, $c = 1$, and $s = \epsilon$, which gives

$$G_1 A = \begin{bmatrix} 1 & \epsilon & 0 \\ -\epsilon & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \epsilon & 0 \\ 0 & \epsilon \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -\epsilon \\ 0 & \epsilon \end{bmatrix}$$

To annihilate the $(3,2)$ entry, $t = -1$, $c = 1/\sqrt{2}$, which gives

$$G_2 G_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -\epsilon \\ 0 & \epsilon \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -\epsilon/\sqrt{2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R} \\ \boldsymbol{O} \end{bmatrix},$$

whose upper 2x2 submatrix $\boldsymbol{R}$ is nonsingular, since $\epsilon \neq 0$.

- **3.26**

(a) $\boldsymbol{G}$ rotates any nonzero 2-vector clockwise by an angle of θ.

(b) Equating $\begin{bmatrix} -c & s \\ s & c \end{bmatrix} = H = I - 2vv^T = \begin{bmatrix} 1 - 2v_1v_1 & -2v_1v_2 \\ -2v_1v_2 & 1 - 2v_1v_1 \end{bmatrix}$. we see that H reflects any nonzero 2-vector through the line defined by the vector $v^{\perp} = \begin{bmatrix} \sqrt{(1-c)/2} & \sqrt{(1+c)/2} \end{bmatrix}^T$.