Sample Solutions of HW of Chapter 7: Interpolations

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Note that, the solutions are for your reference only. If you have any doubts about the correctness of the answers, please let the instructor and the TA know. More importantly, like other math questions, the homework questions may be solved in various ways. Do not assume that the sample solutions here are the only *correct* answers; discuss with others about alternate solutions.

We will not grade your homework assignment, but you are highly encouraged to discuss with us during the Lab hours. The correlation between the homework assignments and quiz/midterm/final questions is high. So you do want to practice more and sooner.

1 Exercises

- **7.1**

  (a) The linear system using the monomial basis is $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

  Solving this system by Gaussian elimination, we obtain $x = [0\ 0\ 1]^T$, so the interpolating polynomial is $p(t) = t^2$.

  (b) The form of a polynomial of degree two using the Lagrange basis is

  $$p(t) = y_1 \frac{(t-t_2)(t-t_3)}{(t_1-t_2)(t_1-t_3)} + y_2 \frac{(t-t_1)(t-t_3)}{(t_2-t_1)(t_2-t_3)} + y_3 \frac{(t-t_1)(t-t_2)}{(t_3-t_1)(t_3-t_2)}.$$

  Substituting the data for this problem, the polynomial becomes

  $$\begin{aligned} p(t) &= 1\frac{(t-0)(t-1)}{(-1-0)(-1-1)} + 0\frac{(t-(-1))(t-1)}{(0-(-1))(0-1)} + 1\frac{(t-(-1))(t-0)}{(1-(-1))(1-0)} \\ &= \frac{t(t-1)}{2} + \frac{t(t+1)}{2}, \end{aligned}$$

  which is equivalent to the polynomial in part (a).

  (c) The linear system using the Newton basis is

  $$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0-(-1) & 0 \\ 1 & 1-(-1) & (1-(-1))(1-0) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Solving this system by forward substitution, we obtain $x = [1 \ -1 \ 1]^T$, so the interpolating polynomial is $p(t) = 1 - (t+1) + (t+1)t$, which again is equivalent to the polynomial in part (a).

- **7.6** We need to plug M, h, and n into the bound $\max_{t\in[t_1,t_n]} |f(t) - p_{n-1}(t)| \leq \frac{Mh^n}{4n}$, where $|f^{(n)}(t)| \leq M$. For this problem $n = 5$, so we need the fifth derivative of $f(t)$, $f^{(5)}(t) = \cos(t)$. The maximum of $\cos(t)$ on $[0, \pi/2]$ occurs at $t = 0$, where $f^{(5)}(0) = 1$, so we can take $M = 1$ and the bound becomes

$$\max_{t\in[t_1,t_n]} |f(t) - p_{n-1}(t)| \leq \frac{1 \cdot (\pi/8)^5}{4 \cdot 5} \leq 4.6695 \times 10^{-4}.$$

The interpolant is $p_4(t) = 0.2871t^4 - 0.20359t^3 + 0.1995t^2 + 0.99632t$. Using the interpolant, $|f(t) - p_{n-1}(t)|$ may be found for any t. To achieve an error bound of 10^{-10},

$$\begin{aligned} \frac{Mh^n}{4n} &= 10^{-10} \\ \frac{1 \cdot (\pi/(2(n-1)))^n}{4n} &= 10^{-10} \\ \frac{\pi^n}{n(2(n-1))^n} &= 4 \times 10^{-10} \\ n &= 10.63. \end{aligned}$$

Therefore, at least 11 points are required to achieve the bound.

- **7.14** To transform the Chebyshev points from $[-1, 1]$ to an arbitrary interval $[a, b]$, we scale by the relative width of the new interval, $(b - a)/2$, and translate by the center of the new interval, $(a + b)/2$, so the transformed Chebyshev points are given by

$$\tilde{t}_i = t_i \frac{b-a}{2} + \frac{a+b}{2}, \quad i = 1, \ldots, k.$$

- **7.16** In keeping with the recursive definition of the B-spline functions B_i^k , all proofs proceed by induction on k.

Property 1:

Base case: By definition, $B_i^0(t) = 0$ for $t < t_i$ or $t > t_{i+1}$.
Inductive hypothesis: $B_i^{k-1}(t) = 0$ for $t < t_i$ or $t > t_{i+k}$.
From the inductive hypothesis, we have

$$\begin{aligned} B_i^{k-1}(t) &= 0 \text{ for } t < t_i \text{ or } t > t_{i+k}, \\ B_{i+1}^{k-1}(t) &= 0 \text{ for } t < t_{i+1} \text{ or } t > t_{i+k+1}, \end{aligned}$$

and hence

$$B_i^k(t) = v_i^k(t) B_i^{k-1}(t) + (1 - v_{i+1}^k(t)) B_{i+1}^{k-1}(t) = 0 \text{ for } t < t_i \text{ or } t > t_{i+k+1}.$$

Property 2:

Base case: By definition, $B_i^0(t) = 1 > 0$ for $t_i < t < t_{i+1}$.
Inductive hypothesis: $B_i^{k-1}(t) > 0$ for $t_i < t < t_{i+1}$.
From the inductive hypothesis, we have

$$v_i^k(t) > 0 \text{ for } t > t_i \Rightarrow v_i^k(t) B_i^{k-1}(t) > 0 \text{ for } t_i < t < t_{i+k},$$
$$(1 - v_{i+1}^k(t)) > 0 \text{ for } t < t_{i+k} \Rightarrow (1 - v_{i+1}^k(t)) B_{i+1}^{k-1}(t) > 0 \text{ for } t_{i+1} < t < t_{i+k+1},$$

and hence

$$B_i^k(t) = v_i^k(t) B_i^{k-1}(t) + (1 - v_{i+1}^k(t)) B_{i+1}^{k-1}(t) > 0 \text{ for } t_i < t < t_{i+k+1}.$$

Property 3:

Base case: By definition, $\sum_{i=-\infty}^{\infty} B_i^0(t) = 1$.
Inductive hypothesis: $\sum_{i=-\infty}^{\infty} B_i^{k-1}(t) = 1$.
From the inductive hypothesis, we have

$$\begin{aligned}
\sum_{i=-\infty}^{\infty} B_i^k(t) &= \sum_{i=-\infty}^{\infty} (v_i^k(t) B_i^{k-1}(t) + (1 - v_{i+1}^k(t)) B_{i+1}^{k-1}(t)) \\
&= \sum_{i=-\infty}^{\infty} v_i^k(t) B_i^{k-1}(t) + \sum_{i=-\infty}^{\infty} (1 - v_{i+1}^k(t)) B_{i+1}^{k-1}(t) \\
&= \sum_{i=-\infty}^{\infty} v_i^k(t) B_i^{k-1}(t) + \sum_{i=-\infty}^{\infty} B_{i+1}^{k-1}(t) - \sum_{i=-\infty}^{\infty} v_{i+1}^k(t) B_{i+1}^{k-1}(t) \\
&= \sum_{i=-\infty}^{\infty} v_i^k(t) B_i^{k-1}(t) + 1 - \sum_{i=-\infty}^{\infty} v_i^k(t) B_i^{k-1}(t) = 1
\end{aligned}$$

Property 4:

Base case: From the definition, B_i^1 is continuous.
Inductive hypothesis: B_i^{k-1} is $k-2$ times continuously differentiable.
As is easily established by a separate induction, for $k-2$ we have

$$\frac{d}{dx} B_i^k(t) = \frac{k}{t_{i+k} - t_i} B_i^{k-1}(t) - \frac{k}{t_{i+k+1} - t_{i+1}} B_{i+1}^{k-1}(t).$$

The derivative of B_i^k is therefore a linear combination of B_i^{k-1} and B_{i+1}^{k-1}, which by the inductive hypothesis are both continuously differentiable $k-2$ times. Thus, B_i^k is continuously differentiable $k-1$ times.

Property 5:

Base case: B_i^1 is obviously linearly independent.
Inductive hypothesis: $B_{1-k}^{k-1}, \ldots, B_{n-1}^{k-1}$ are linearly independent.
If f is a linear combination of $B_{1-k}^k, \ldots, B_{n-1}^k$ such that $f = 0$, then we would also

have $f' = 0$. But we saw in the proof of Property 4 that the derivative of each B_i^k is a linear combination of B_i^{k-1} and B_{i+1}^{k-1}. Since $B_{1-k}^{k-1}, \ldots, B_{n-1}^{k-1}$ are linearly independent by the inductive hypothesis, this implies that there can be no such f that is a nontrivial linear combination of $B_{1-k}^k, \ldots, B_{n-1}^k$, and hence the latter are linearly independent.

Property 6:

This follows from the linear independence of $B_{1-k}^k, \ldots, B_{n-1}^k$ and the dimensionality of the spaces involved.