Sample Solutions of HW of Chapter 6: Optimization

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Note that, the solutions are for your reference only. If you have any doubts about the correctness of the answers, please let the instructor and the TA know. More importantly, like other math questions, the homework questions may be solved in various ways. Do not assume that the sample solutions here are the only *correct* answers; discuss with others about alternate solutions.

We will not grade your homework assignment, but you are highly encouraged to discuss with us during the Lab hours. The correlation between the homework assignments and quiz/midterm/final questions is high. So you do want to practice more and sooner.

1 Exercises

- **6.5**

(a) $\nabla f(x,y) = \begin{bmatrix} 2x-4y \\ 2y-4x \end{bmatrix}$, $\boldsymbol{H}_f(x,y) = \begin{bmatrix} 2 & -4 \\ -4 & 2 \end{bmatrix}$. Critical point $(0,0)$ is a saddle point because $\boldsymbol{H}_f(0,0)$ is indefinite. There is no global minimum or maximum.

(b) $\nabla f(x,y) = \begin{bmatrix} 4x^3-4y \\ -4x+4y^3 \end{bmatrix}$, $\boldsymbol{H}_f(x,y) = \begin{bmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{bmatrix}$. Critical point $(0,0)$ is a saddle point because $\boldsymbol{H}_f(0,0)$ is indefinite. Critical point $(1,1)$ is a local minimum because $\boldsymbol{H}_f(1,1)$ is positive definite. Critical point $(-1,-1)$ is a local minimum because $\boldsymbol{H}_f(-1,-1)$ is positive definite. Both local minima are global minima, but there is no global maximum.

(c)

$$\nabla f(x,y) = \begin{bmatrix} 6x^2-12xy+6y^2-6x+6y \\ -6x^2+12xy+6x \end{bmatrix},$$

$$\boldsymbol{H}_f(x,y) = \begin{bmatrix} 12x-12y-6 & -12x+12y+6 \\ -12x+12y+6 & 12x \end{bmatrix}.$$

Critical point $(0,0)$ is a saddle point because $\boldsymbol{H}_f(0,0)$ is indfinite. Critical point $(1,0)$ is a local minimum because $\boldsymbol{H}_f(1,0)$ is positive definite. Critical point $(0,-1)$ is a saddle point because $\boldsymbol{H}_f(0,-1)$ is indefinite. There is no global minimum or maximum.

(d)

$$\nabla f(x,y) = \begin{bmatrix} 4(x-y)^3 + 2x - 2 \\ -4(x-y)^3 - 2y + 2 \end{bmatrix},$$

$$\boldsymbol{H}_f(x,y) = \begin{bmatrix} 12(x-y)^2 + 2 & -12(x-y)^2 \\ -12(x-y)^2 & 12(x-y)^2 - 2 \end{bmatrix}.$$

Critical point $(1,1)$ is saddle point because $\boldsymbol{H}_f(1,1)$ is indefinite. There is no global minimum or maximum.

- **6.6**

(a)

$$\nabla f(x,y) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}, \boldsymbol{J}_g(x,y) = \begin{bmatrix} 1 & 1 \end{bmatrix}, \nabla \mathcal{L}(x,y,\lambda) = \begin{bmatrix} 2x + \lambda \\ 2y + \lambda \\ x + y - 1 \end{bmatrix},$$

$$\boldsymbol{B}(x,y,\lambda) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \boldsymbol{z}(x,y) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Critical point $(0.5, 0.5, -1)$ is a constrained minimum because $\boldsymbol{z}^T\boldsymbol{B}\boldsymbol{z} = 4 > 0$.

(b)

$$\nabla f(x,y) = \begin{bmatrix} 3x^2 \\ 3y^2 \end{bmatrix}, \boldsymbol{J}_g(x,y) = \begin{bmatrix} 1 & 1 \end{bmatrix}, \nabla \mathcal{L}(x,y,\lambda) = \begin{bmatrix} 3x^2 + \lambda \\ 3y^2 + \lambda \\ x + y - 1 \end{bmatrix},$$

$$\boldsymbol{B}(x,y,\lambda) = \begin{bmatrix} 6x & 0 \\ 0 & 6y \end{bmatrix}, \boldsymbol{z}(x,y) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Critical point $(0.5, 0.5, -0.75)$ is a constrained minimum because $\boldsymbol{z}^T\boldsymbol{B}\boldsymbol{z} = 4 > 0$.

(c)

$$\nabla f(x,y) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \boldsymbol{J}_g(x,y) = \begin{bmatrix} 2x & 2y \end{bmatrix}, \nabla \mathcal{L}(x,y,\lambda) = \begin{bmatrix} 2 + 2\lambda x \\ 1 + 2\lambda y \\ x^2 + y^2 - 1 \end{bmatrix},$$

$$\boldsymbol{B}(x,y,\lambda) = \begin{bmatrix} 2\lambda & 0 \\ 0 & 2\lambda \end{bmatrix}, \boldsymbol{z}(x,y) = \begin{bmatrix} y \\ -x \end{bmatrix}.$$

Critical point $(-0.894, -0.447, 1.12)$ is a constrained minimum because $\boldsymbol{z}^T\boldsymbol{B}\boldsymbol{z} = 2.24 > 0$. Critical point $(0.894, 0.447, -1.12)$ is a constrained maximum because $\boldsymbol{z}^T\boldsymbol{B}\boldsymbol{z} = -2.24 < 0$.

(d)

$$\nabla f(x,y) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}, \boldsymbol{J}_g(x,y) = \begin{bmatrix} y^2 & 2xy \end{bmatrix}, \nabla \mathcal{L}(x,y,\lambda) = \begin{bmatrix} 2x + \lambda y^2 \\ 2y + 2\lambda xy \\ xy^2 - 1 \end{bmatrix},$$

$$\boldsymbol{B}(x,y,\lambda) = \begin{bmatrix} 2 & 2\lambda y \\ 2\lambda y & 2+2\lambda x \end{bmatrix}, \boldsymbol{z}(x,y) = \begin{bmatrix} 2x \\ -y \end{bmatrix}.$$

Critical point $(0.794, 1.12, -1.26)$ is a constrained minimum because $\boldsymbol{z}^T\boldsymbol{B}\boldsymbol{z} = 15.1 > 0$.

- **6.7**

$$\nabla f(\boldsymbol{x}) = \begin{bmatrix} 2x_1 - 2 \\ 2x_2 \\ -2x_3 + 4 \end{bmatrix}, \boldsymbol{J}_g(\boldsymbol{x}) = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}, \nabla\mathcal{L}(\boldsymbol{x},\lambda) = \begin{bmatrix} 2x_1 - 2 + \lambda \\ 2x_2 - \lambda \\ -2x_3 + 4 + 2\lambda \\ x_1 - x_2 + 2x_3 - 2 \end{bmatrix},$$

$$\boldsymbol{B}(\boldsymbol{x},\lambda) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \boldsymbol{Z}(\boldsymbol{x}) = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}, \boldsymbol{Z}^T\boldsymbol{B}\boldsymbol{Z} = \begin{bmatrix} 4 & 4 \\ 4 & 6 \end{bmatrix}.$$

Critical point $(2.5, -1.5, -1, -3)$ is a constrained minimum because $\boldsymbol{Z}^T\boldsymbol{B}\boldsymbol{Z}$ is positive definite.

- **6.12**

  (a) Suppose that $\boldsymbol{x}$ is a local minimum of a convex function f on a convex set $S \in \mathbb{R}^n$, but not a global minimum. Then there exists a point $\boldsymbol{y} \in S$ such that $f(\boldsymbol{y}) < f(\boldsymbol{x})$. Consider the line segment between $\boldsymbol{x}$ and $\boldsymbol{y}$, and recall that the graph of a convex function along any line segment in S must lie on or below the chord connecting the function values at the endpoints of the segment. But this contradicts the assumption that $\boldsymbol{x}$ is a local minimum. Thus, any local minimum of f must be a global minimum.

  (b) Suppose that $\boldsymbol{x}$ and $\boldsymbol{y}$ are distinct local minima of a strictly convex function f on a convex set $S \in \mathbb{R}^n$. Consider the line segment between $\boldsymbol{x}$ and $\boldsymbol{y}$, and recall that the graph of a strictly convex function along any line segment in S must lie strictly below the chord connecting the function values at the endpoints of the segment. But this contradicts the assumption that $\boldsymbol{x}$ and $\boldsymbol{y}$ are local minima. Thus, any local minimum of f must be the unique global minimum.

- **6.17**

  (a) There are five vertices: $(0, 0)$, $(0, 1.5)$, $(0.857, 0.857)$, $(1.09, 0.545)$, and $(1.2, 0)$.

  (b) The corresponding values of the objective function are 0, -3, -4.29, -4.37, and -3.6, respectively, so the minimum occurs at $(1.09, 0.545)$.