Department of Computer Science
National Tsing Hua University

CS 5244: Introduction to Cyber Physical Systems

Unit 2: Model-Based Design (Ch. 2)

Instructor: Cheng-Hsin Hsu

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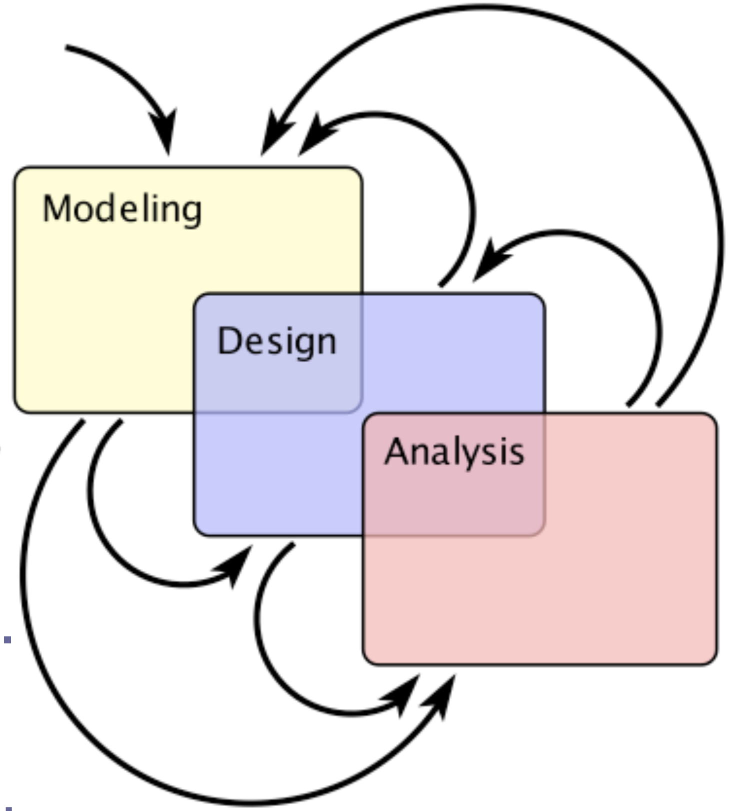
Modeling, Design, Analysis



Modeling is the process of gaining a deeper understanding of a system through imitation. Models specify **what** a system does.

Design is the structured creation of artifacts. It specifies **how** a system does what it does. This includes optimization.

Analysis is the process of gaining a deeper understanding of a system through dissection. It specifies **why** a system does what it does (or fails to do what a model says it should do).

What is Modeling?

Developing insight about a system, process, or artifact through imitation.

A *model* is the artifact that imitates the system, process, or artifact of interest.

A *mathematical model* is model in the form of a set of definitions and mathematical formulas.

What is Model-Based Design?

1. Create a mathematical model of all the parts of the cyber-physical system
   - Physical world
   - Control system
   - Software environment
   - Hardware platform
   - Network
   - Sensors and actuators
2. Construct the implementation from the model
   - Construction may be automated, like a compiler
   - More commonly, some parts are automatically constructed

Modeling Techniques in this Course

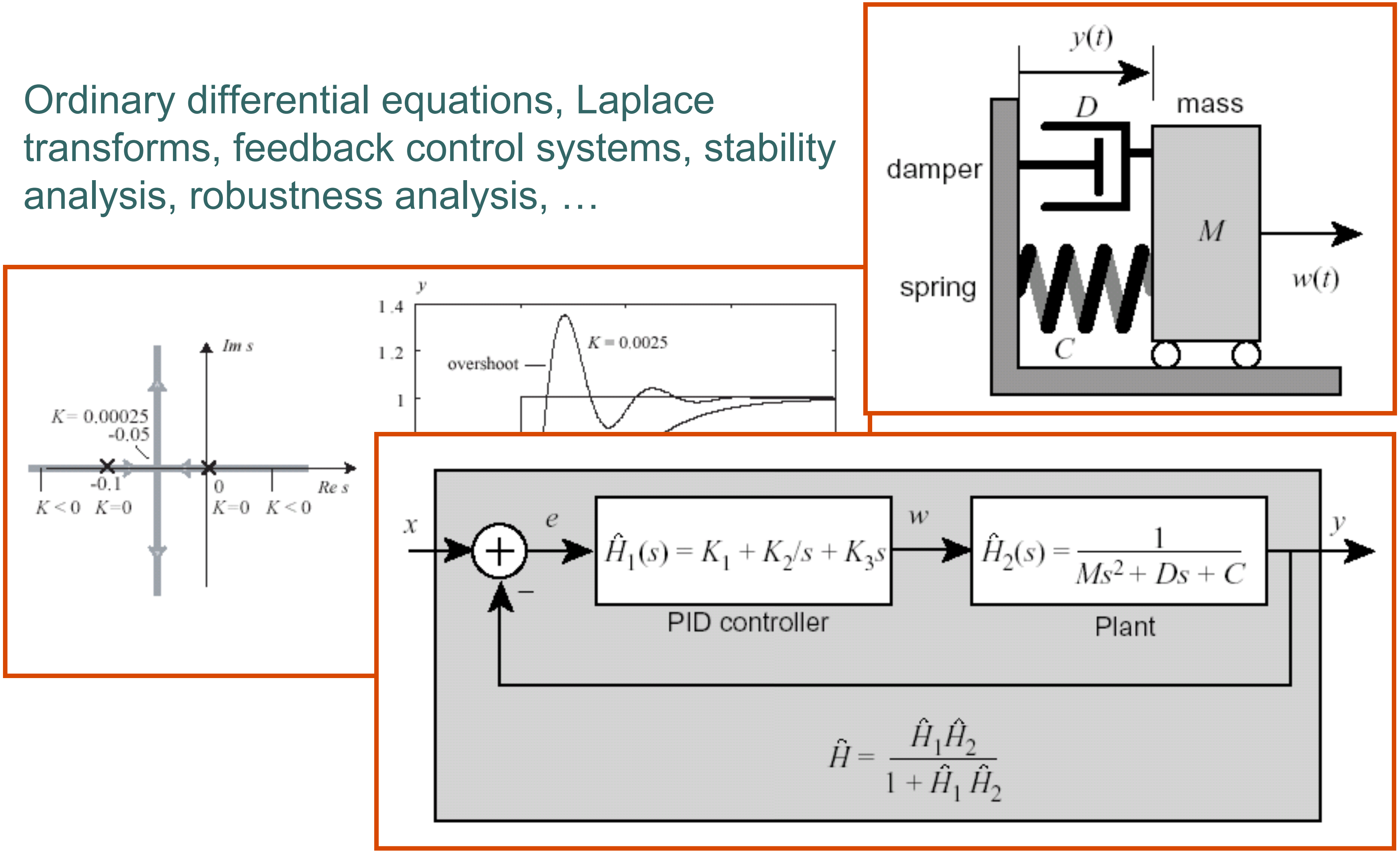
Models that are abstractions of **system dynamics** (how things change over time)

Examples:

- Modeling physical phenomena – ODEs
- Feedback control systems – time-domain modeling
- Modeling modal behavior – FSMs, hybrid automata
- Modeling sensors and actuators – calibration, noise
- Modeling software – concurrency, real-time models
- Modeling networks – latencies, error rates, packet loss

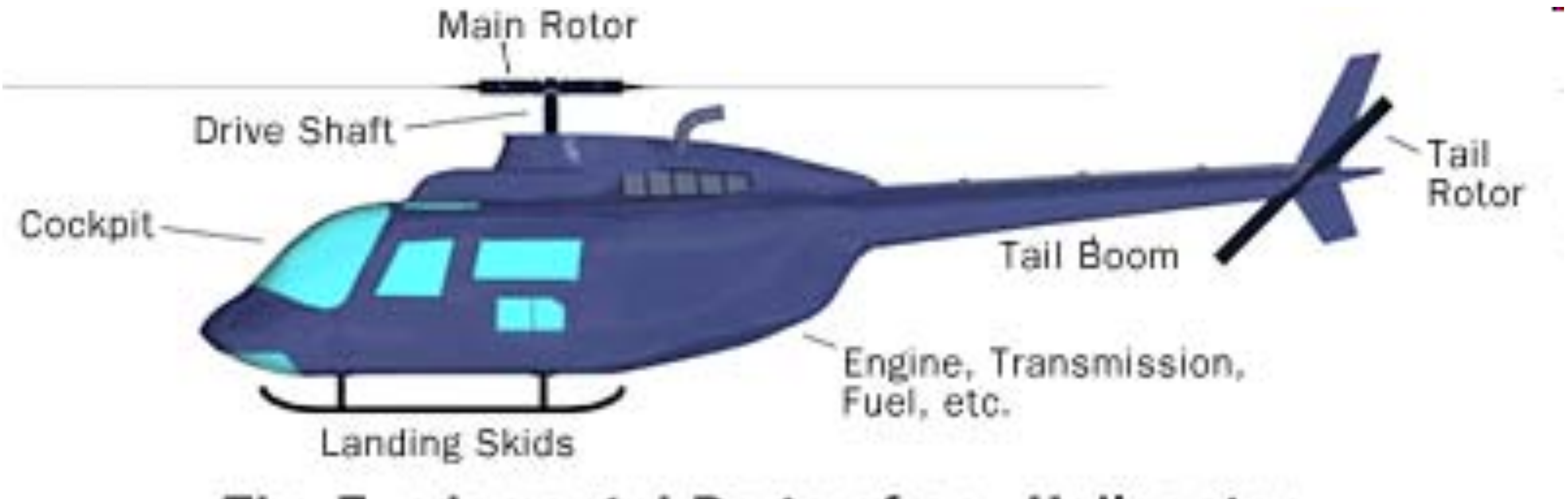
Modeling of Continuous Dynamics

Ordinary differential equations, Laplace transforms, feedback control systems, stability analysis, robustness analysis, …



Im s
Re s
K = 0.00025
-0.05
-0.1
0
K < 0
K = 0
K = 0
K < 0
y
1.4
1.2
1
overshoot
K = 0.0025
y(t)
D
mass
damper
M
spring
w(t)
C
x
e
w
y
$\hat{H}_1(s) = K_1 + K_2/s + K_3 s$
$\hat{H}_2(s) = \frac{1}{Ms^2 + Ds + C}$
PID controller
Plant
−
$\hat{H} = \frac{\hat{H}_1 \hat{H}_2}{1 + \hat{H}_1 \hat{H}_2}$

An Example: Modeling Helicopter Dynamics



The Fundamental Parts of any Helicopter

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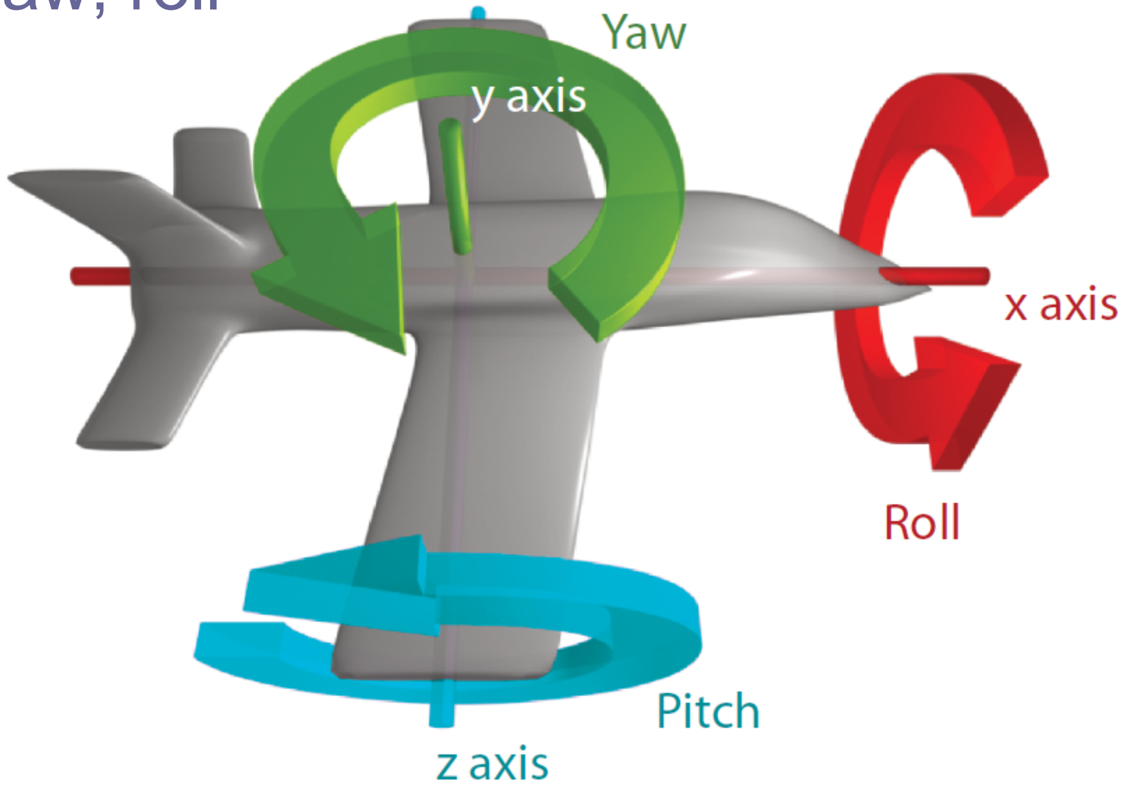
Modeling Physical Motion

Six degrees of freedom:

- Position: x, y, z
- Orientation: pitch, yaw, roll



Notation

Position is given by three functions:

$$x\colon \mathbb{R} \to \mathbb{R}$$
$$y\colon \mathbb{R} \to \mathbb{R}$$
$$z\colon \mathbb{R} \to \mathbb{R}$$

where the domain $\mathbb{R}$ represents time and the co-domain (range) $\mathbb{R}$ represents position along the axis. Collecting into a vector:

$$\mathbf{x}\colon \mathbb{R} \to \mathbb{R}^3$$

Position at time $t \in \mathbb{R}$ is $\mathbf{x}(t) \in \mathbb{R}^3$.

Notation

Velocity

$$\dot{\mathbf{x}} \colon \mathbb{R} \to \mathbb{R}^3$$

is the derivative, $\forall\, t \in \mathbb{R}$,

$$\dot{\mathbf{x}}(t) = \frac{d}{dt}\mathbf{x}(t)$$

Acceleration $\ddot{\mathbf{x}} \colon \mathbb{R} \to \mathbb{R}^3$ is the second derivative,

$$\ddot{\mathbf{x}} = \frac{d^2}{dt^2}\mathbf{x}$$

Force on an object is $\mathbf{F} \colon \mathbb{R} \to \mathbb{R}^3$.

Newton’s Second Law

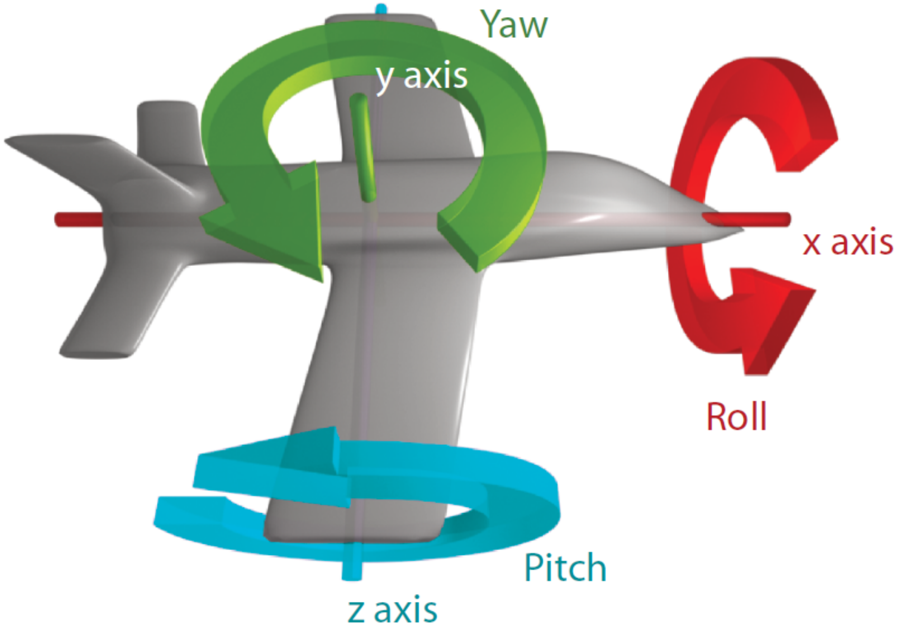
Newton’s second law states $\forall\, t \in \mathbb{R}$,

$$\mathbf{F}(t) = M\ddot{\mathbf{x}}(t)$$

where M is the mass. To account for initial position and velocity, convert this to an integral equation

$$\begin{aligned}\mathbf{x}(t) &= \mathbf{x}(0) + \int\limits_0^t \dot{\mathbf{x}}(\tau)d\tau \\ &= \mathbf{x}(0) + t\dot{\mathbf{x}}(0) + \frac{1}{M}\int\limits_0^t\int\limits_0^\tau \mathbf{F}(\alpha)d\alpha d\tau,\end{aligned}$$

Orientation

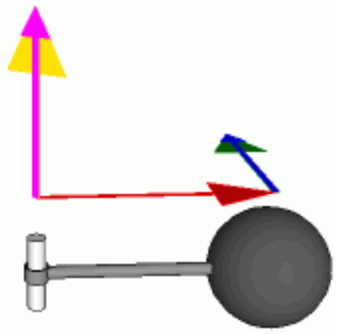


- Orientation: $\theta : \mathbb{R} \to \mathbb{R}^3$
- Angular velocity: $\dot{\theta} : \mathbb{R} \to \mathbb{R}^3$
- Angular acceleration: $\ddot{\theta} : \mathbb{R} \to \mathbb{R}^3$
- Torque: $\mathbf{T} : \mathbb{R} \to \mathbb{R}^3$

$$\theta(t) = \begin{bmatrix} \theta_x(t) \\ \theta_y(t) \\ \theta_z(t) \end{bmatrix} = \begin{bmatrix} \text{roll} \\ \text{yaw} \\ \text{pitch} \end{bmatrix}$$

Angular version of force is torque.
For a point mass rotating around a fixed axis:

- radius of the arm: $r \in \mathbb{R}$
- force orthogonal to arm: $f \in \mathbb{R}$
- mass of the object: $m \in \mathbb{R}$



$$T_y(t) = r\,f(t)$$

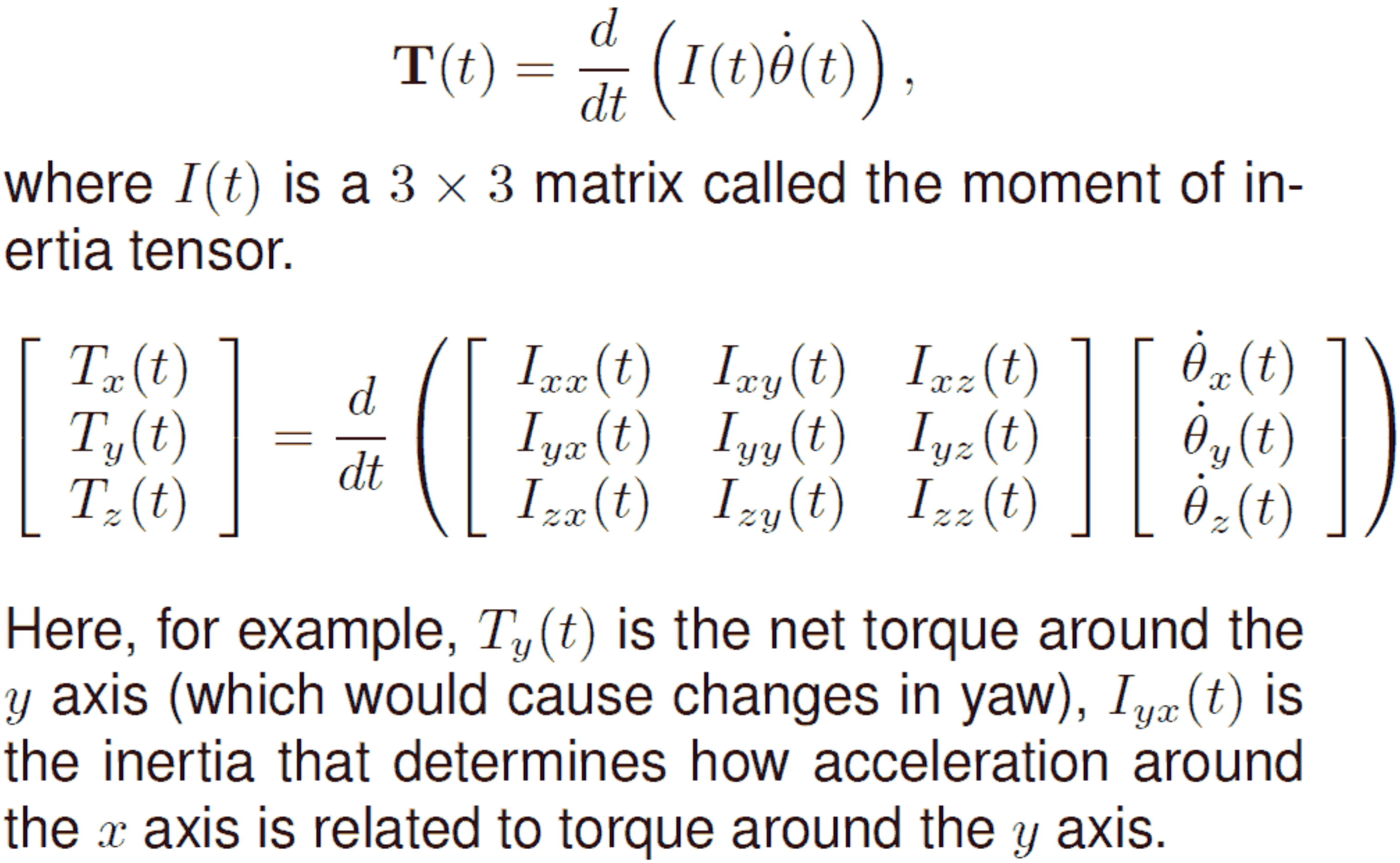
angular momentum, momentum

Just as force is a push or a pull, a torque is a twist.

Units: newton-meters/radian, Joules/radian

Note that radians are meters/meter (2π meters of circumference per 1 meter of radius), so as units, are optional.

Rotational Version of Newton’s Second Law



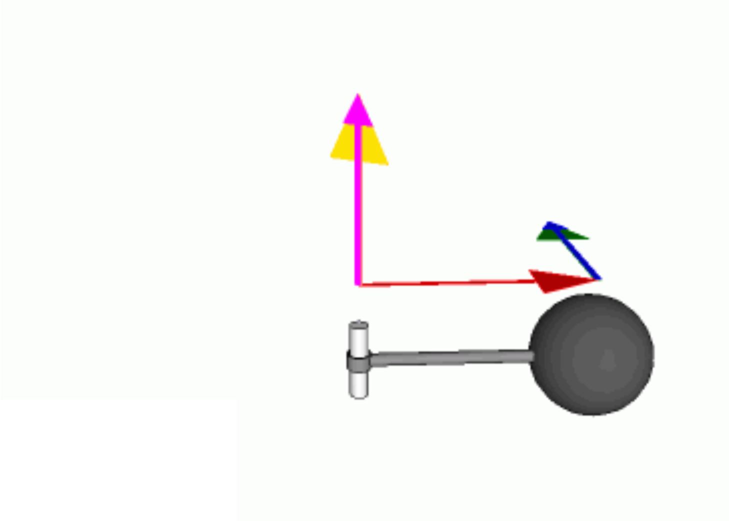
$$\mathbf{T}(t) = \frac{d}{dt}\left(I(t)\dot{\theta}(t)\right),$$

where $I(t)$ is a 3×3 matrix called the moment of inertia tensor.

$$\begin{bmatrix} T_x(t) \\ T_y(t) \\ T_z(t) \end{bmatrix} = \frac{d}{dt}\left(\begin{bmatrix} I_{xx}(t) & I_{xy}(t) & I_{xz}(t) \\ I_{yx}(t) & I_{yy}(t) & I_{yz}(t) \\ I_{zx}(t) & I_{zy}(t) & I_{zz}(t) \end{bmatrix}\begin{bmatrix} \dot{\theta}_x(t) \\ \dot{\theta}_y(t) \\ \dot{\theta}_z(t) \end{bmatrix}\right)$$

Here, for example, $T_y(t)$ is the net torque around the y axis (which would cause changes in yaw), $I_{yx}(t)$ is the inertia that determines how acceleration around the x axis is related to torque around the y axis.

Simple Example



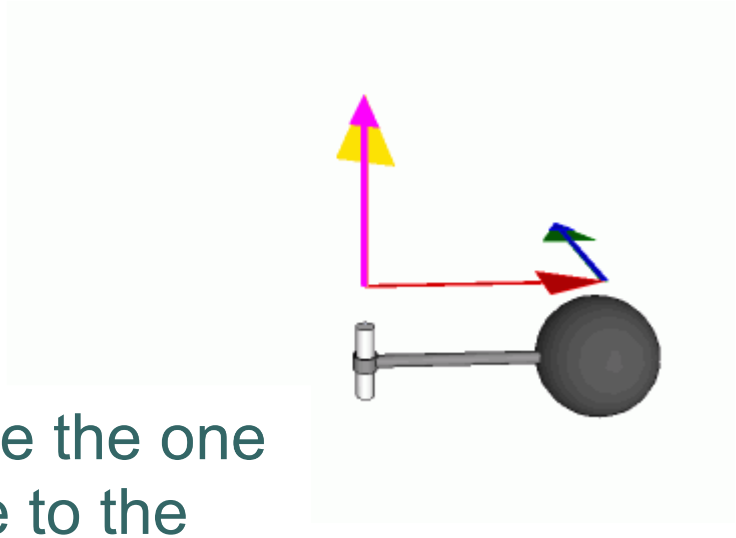
Yaw dynamics:

$$T_y(t) = I_{yy}\ddot{\theta}_y(t)$$

To account for initial angular velocity, write as

$$\dot{\theta}_y(t) = \dot{\theta}_y(0) + \frac{1}{I_{yy}}\int_0^t T_y(\tau)d\tau.$$

Feedback Control Problem

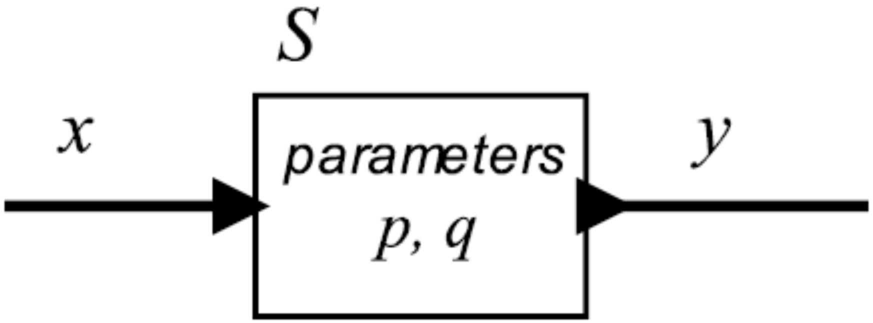


A helicopter without a tail rotor, like the one below, will spin uncontrollably due to the torque induced by friction in the rotor shaft.

Control system problem:
Apply torque using the tail rotor to counterbalance the torque of the top rotor.

Actor Model of Systems

A *system* is a function that accepts an input *signal* and yields an output signal.



The domain and range of the system function are sets of signals, which themselves are functions.

$$x : \mathbb{R} \to \mathbb{R}, \quad y : \mathbb{R} \to \mathbb{R}$$

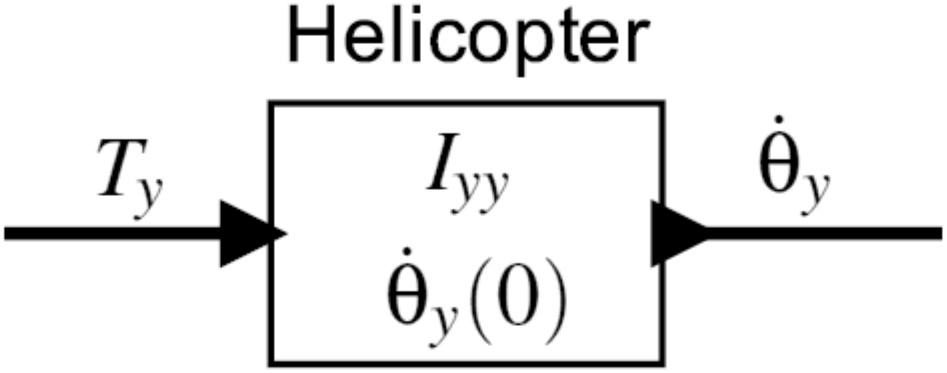
$$S : X \to Y$$

$$X = Y = (\mathbb{R} \to \mathbb{R})$$

Parameters may affect the definition of the function *S*.

Actor model of the helicopter

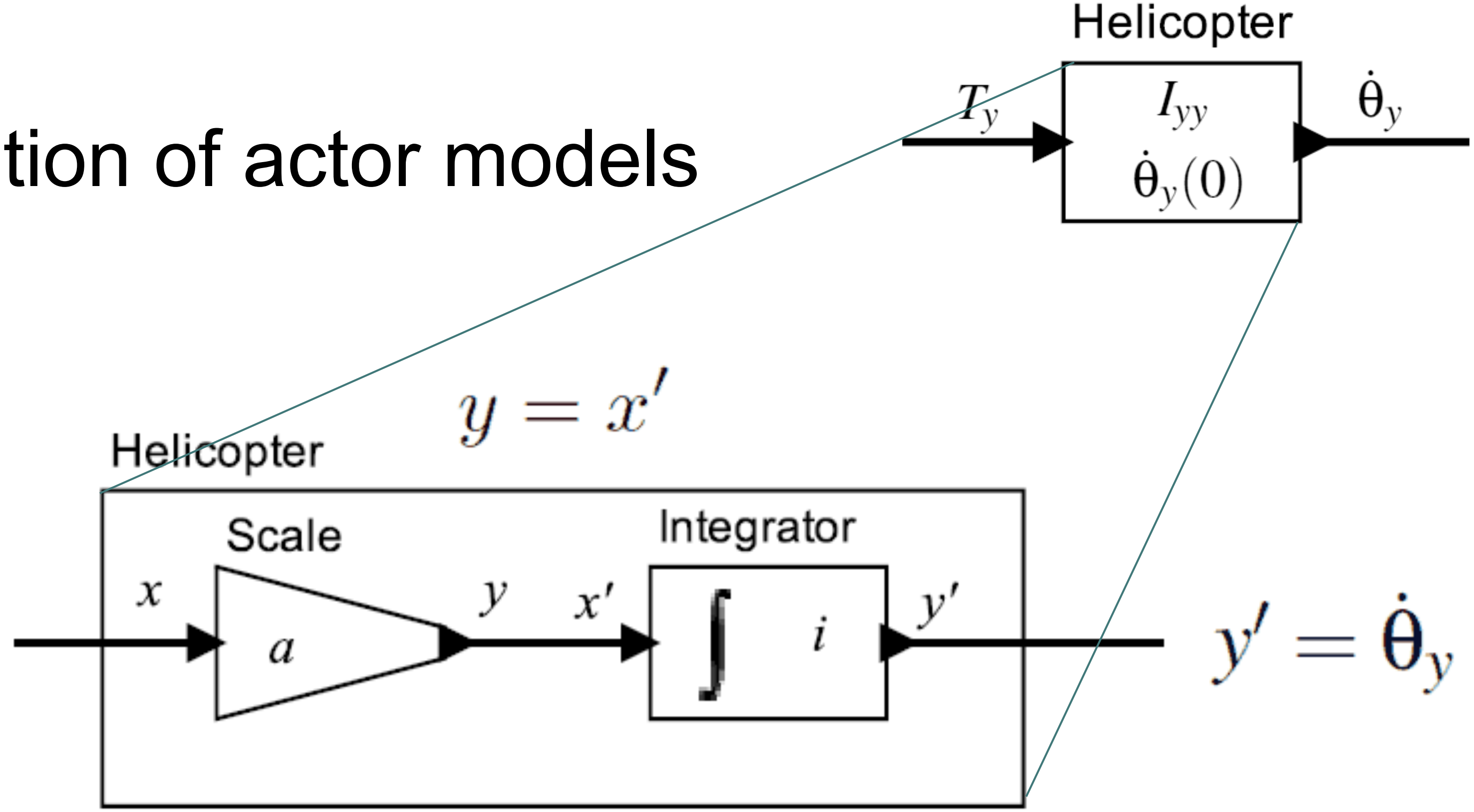
Input is the net torque of the tail rotor and the top rotor. Output is the angular velocity around the *y* axis.



Parameters of the model are shown in the box. The input and output relation is given by the equation to the right.

$$\dot{\theta}_y(t) = \dot{\theta}_y(0) + \frac{1}{I_{yy}} \int\limits_0^t T_y(\tau) d\tau$$

Composition of actor models



$$y = x'$$

$$x = T_y$$

$$y' = \dot{\theta}_y$$

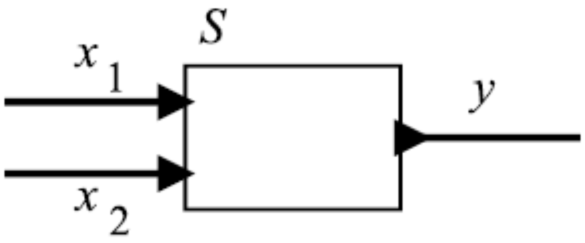
$$\forall\, t \in \mathbb{R}, \quad y(t) = ax(t) \qquad y'(t) = i + \int_0^t x'(\tau)d\tau$$

$$y = ax$$

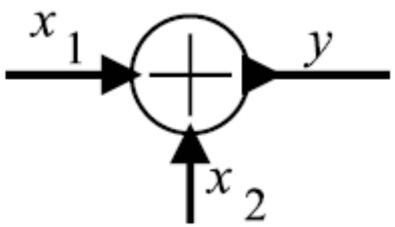
$$a = 1/I_{yy}$$

$$i = \dot{\theta}_y(0)$$

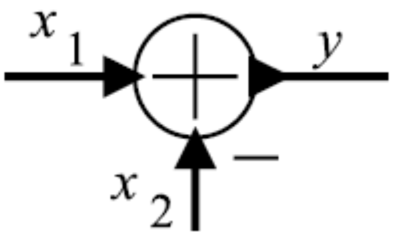
Actor models with multiple inputs



$$S\colon (\mathbb{R} \to \mathbb{R})^2 \to (\mathbb{R} \to \mathbb{R})$$

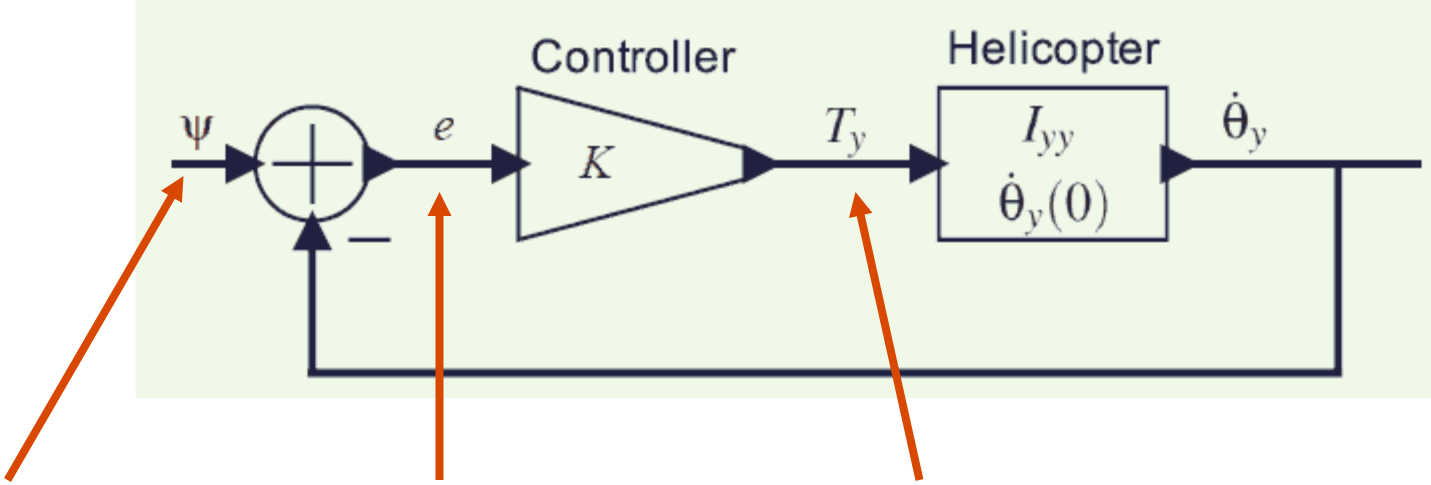


$$\forall\, t \in \mathbb{R}, \quad y(t) = x_1(t) + x_2(t)$$



$$(S(x_1, x_2))(t) = y(t) = x_1(t) - x_2(t)$$

Proportional controller



desired
angular
velocity

error
signal

net
torque

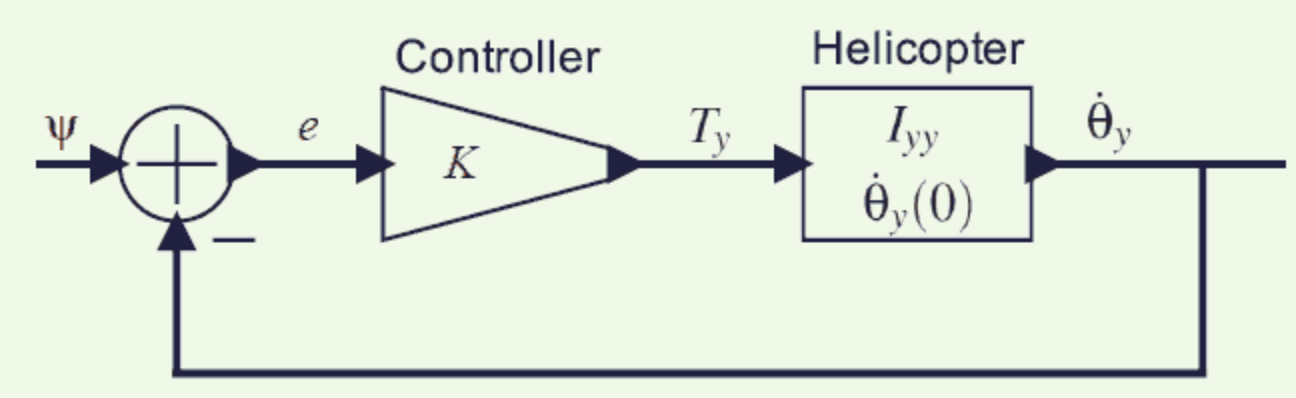
$$e(t) = \psi(t) - \dot{\theta}_y(t)$$

$$T_y(t) = Ke(t)$$

$$\begin{aligned}\dot{\theta}_y(t) &= \dot{\theta}_y(0) + \frac{1}{I_{yy}} \int_0^t T_y(\tau) d\tau \\ &= \dot{\theta}_y(0) + \frac{K}{I_{yy}} \int_0^t (\psi(\tau) - \dot{\theta}_y(\tau)) d\tau\end{aligned}$$

Note that the angular velocity appears on both sides, so this equation is not trivial to solve.

Behavior of the controller



$$\dot\theta_y(t) = \dot\theta_y(0) + \frac{K}{I_{yy}} \int_0^t (\psi(\tau) - \dot\theta_y(\tau))d\tau$$

Assume that helicopter is initially at rest,

$$\dot\theta(0) = 0,$$

and that the desired signal is

$$\psi(t) = au(t)$$

for some constant a.
By calculus (see notes), the solution is

$$\dot\theta_y(t) = au(t)(1 - e^{-Kt/I_{yy}})$$

Exercise

Reformulate the helicopter model so that it has two inputs, the torque of the top rotor and the torque of the tail rotor.

Show (by simulation) that if the top rotor applies a constant torque, then our controller cannot keep the helicopter from rotating. Increasing the feedback gain, however, reduces the rate of rotation.

A better controller would include an integrator in the controller. Such controllers are studied in control systems course.

Questions

- How do we measure the angular velocity?
- Can this controller be implemented in software?
- How does the behavior change when it is implemented in software?

Other Modeling Techniques we will talk about

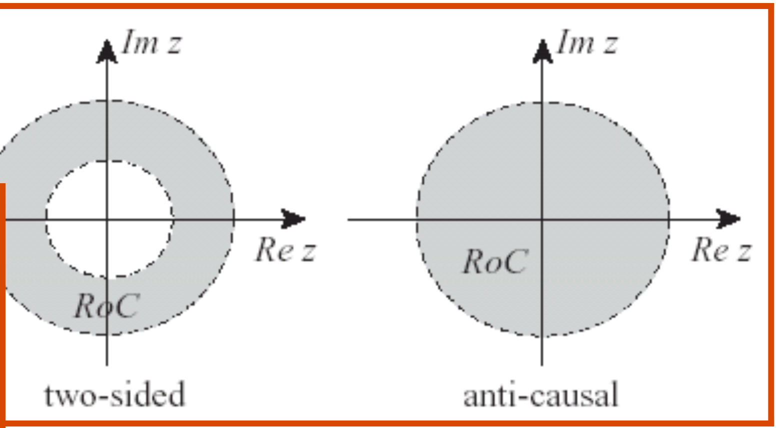
- State machines
  - sequential decision logic
- Synchronous/reactive concurrent composition
  - concurrent computation
  - composes well with state machines
- Dataflow models
  - exploitable parallelism
  - well suited to signal processing
- Discrete-event models
  - explicit about time
- Time-driven
  - suitable for periodic, timed actions
- Continuous-time models
  - models of physical dynamics
  - extended to “hybrid systems” to embrace computation

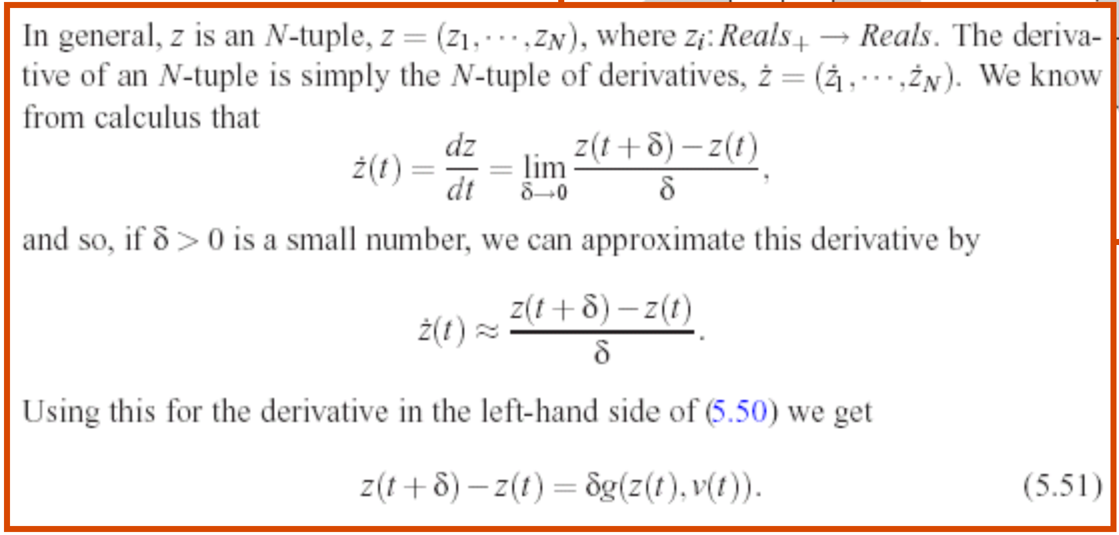
Discretized Model
A Step Towards Software

Numerical integration techniques provided sophisticated ways to get from the continuous idealizations to computable algorithms.

Discrete-time signal processing techniques offer the same sophisticated stability analysis as continuous-time methods.

But it’s not accurate for software controllers (fails on correctness)





In general, z is an N-tuple, $z = (z_1, \cdots, z_N)$, where $z_i\colon Reals_+ \to Reals$. The derivative of an N-tuple is simply the N-tuple of derivatives, $\dot{z} = (\dot{z}_1, \cdots, \dot{z}_N)$. We know from calculus that

$$\dot{z}(t) = \frac{dz}{dt} = \lim_{\delta \to 0} \frac{z(t+\delta) - z(t)}{\delta},$$

and so, if $\delta > 0$ is a small number, we can approximate this derivative by

$$\dot{z}(t) \approx \frac{z(t+\delta) - z(t)}{\delta}.$$

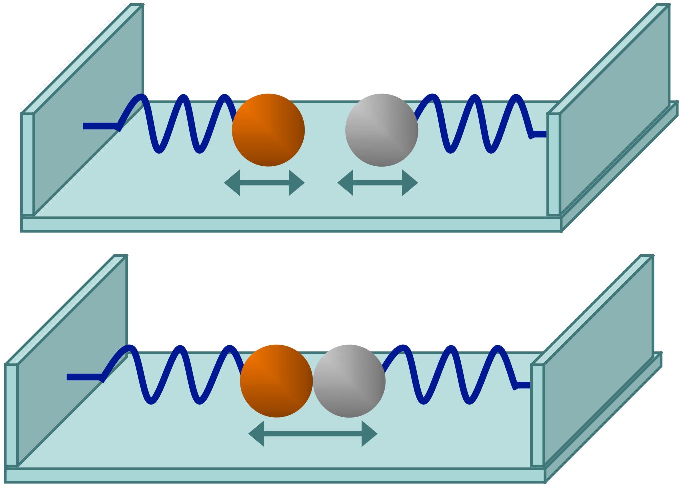
Using this for the derivative in the left-hand side of (5.50) we get

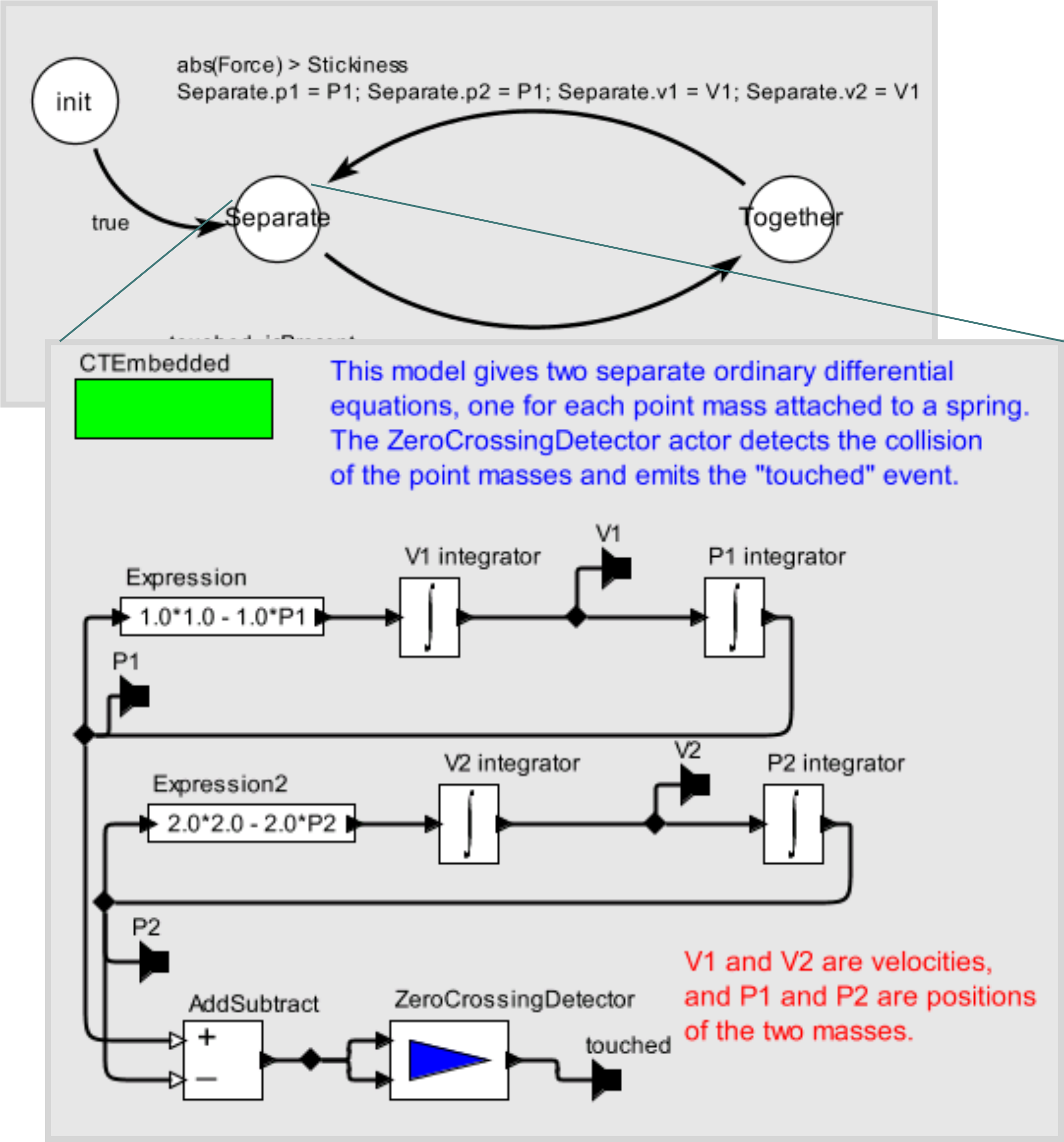
$$z(t+\delta) - z(t) = \delta g(z(t), v(t)). \qquad (5.51)$$

Hybrid Systems –
Union of Continuous & Discrete

A good starting point, but has limitations.

E.g. Consider building a hybrid system model for software running under a multitasking real-time OS.





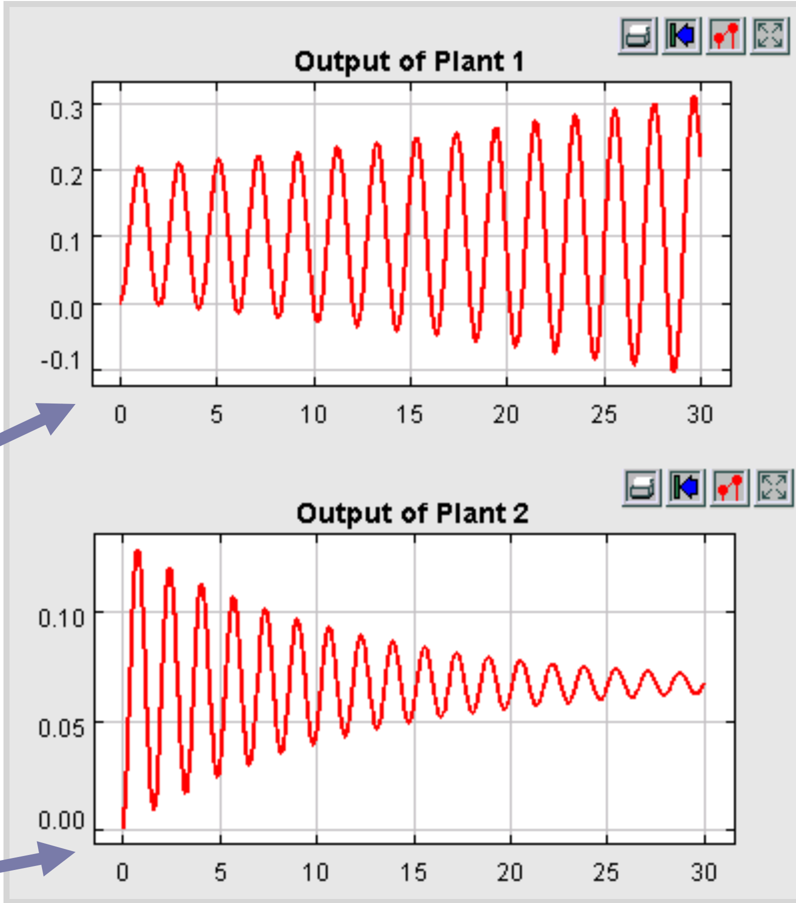
This model gives two separate ordinary differential equations, one for each point mass attached to a spring. The ZeroCrossingDetector actor detects the collision of the point masses and emits the "touched" event.

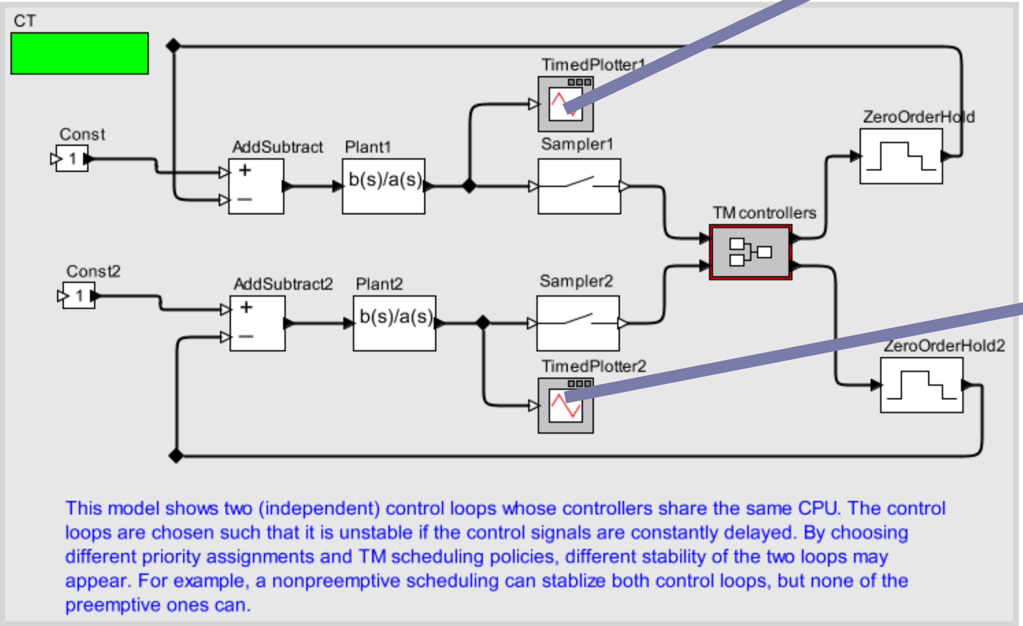
V1 and V2 are velocities, and P1 and P2 are positions of the two masses.

Understanding models can be very challenging

An example, due to Jie Liu, has two controllers sharing a CPU under an RTOS. Under preemptive multitasking, only one can be made stable (depending on the relative priorities). Under non-preemptive multitasking, both can be made stable.

Theory for this is lacking, so designers resort to simulation and testing.





This model shows two (independent) control loops whose controllers share the same CPU. The control loops are chosen such that it is unstable if the control signals are constantly delayed. By choosing different priority assignments and TM scheduling policies, different stability of the two loops may appear. For example, a nonpreemptive scheduling can stablize both control loops, but none of the preemptive ones can.

Key Concepts in Model-Based Design

- Models describe physical dynamics.
- Specifications are executable models.
- Models are composed to form designs.
- Models evolve during design.
- Deployed code may be (partially) generated from models.
- Modeling languages have semantics.
- Modeling languages themselves may be modeled (meta models)

For embedded systems, this is about

- Time
- Concurrency
- Dynamics