CS 2336: Discrete Mathematics

Chapter 7

Relations: The Second Time Around

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Outline

Reviews

- For sets A, B, any subset of $A \times B$ is called a (binary) relation from A to B. Any subset of $A \times A$ is called a (binary) relation on A
  - Ex: Let Σ be an alphabet, with language $A \subseteq \Sigma^*$. For x, y in A, we define $x\mathscr{R}y$ if x is a prefix of y.
  - Ex: Consider a state machine $M = (S, \mathscr{I}, \mathscr{O}, \nu, \omega)$
    - First level of reachability: $s_1\mathscr{R}s_2$ if $\nu(s_1, x) = s_2$
    - Second level: $s_1\mathscr{R}s_2$ if $\nu(s_1, x_1x_2) = s_2, x_1x_2 \in \mathscr{I}^2$

Reflexive

- A relation $\mathscr{R}$ on a set A is called reflexive if for all

  $x \in A, (x, x) \in \mathscr{R}$

- Ex 7.4: Consider $A=\{1,2,3,4\}$, a relation $\mathscr{R} \subseteq A \times A$ is reflexive iff $\mathscr{R} \supseteq \{(1,1),(2,2),(3,3),(4,4)\}$
  - Are the following relations reflexive?

    $\mathscr{R}_1 = \{(1,1),(2,2),(2,3)\}$

    $\mathscr{R}_2 = \{(x,y)|x,y \in A, x \geqslant y\}$

Symmetric

- A relation $\mathscr{R}$ on a set *A* is called symmetric if for all $x, y \in A$, we know $(x,y) \in \mathscr{R} \implies (y,x) \in \mathscr{R}$

- Ex 7.6: Consider *A*={*1*,*2*,*3*}, are the following relations symmetric or reflexive?

$$\mathscr{R}_1 = \{(1,2),(2,1),(1,3),(3,1)\}$$
$$\mathscr{R}_2 = \{(1,1),(2,2),(2,3),(3,3)\}$$
$$\mathscr{R}_3 = \{(1,1),(2,2),(2,3),(3,3),(3,2)\}$$

Transitive

- A relation $\mathscr{R}$ on a set A is called transitive if for all $x, y, z \in A$, we know $x \mathscr{R} y$ and $y \mathscr{R} z \Longrightarrow x \mathscr{R} z$

- Ex 7.10: Consider $A=\{1,2,3,4\}$, are the following relations transitive?

$$\mathscr{R}_1 = \{(1,1),(2,3),(3,4),(2,4)\}$$
$$\mathscr{R}_2 = \{(1,3),(3,4)\}$$

Antisymmetric

- A relation $\mathscr{R}$ on a set A is called antisymmetric if for all $a, b \in A$, if $a\mathscr{R}b$ and $b\mathscr{R}a \implies a = b$

- Ex 7.11: For any universe $\mathscr{U}$, relation $\mathscr{R}$ defined on $\mathscr{P}(\mathscr{U})$ by $(A, B) \in \mathscr{R}$ if $A \subseteq B$. Is this relation antisymmetric? Howa bout reflexive, symmetric, and transtive?

Partial Order

- A relation $\mathscr{R}$ on a set A is called partial order if it is reflexive, antisymmetric, and transitive

- Ex 7.14: Are the following relations partial order?
  - Define a relation on $\mathbb{Z}$ by $(a, b) \in \mathscr{R}$ if $a \leqslant b$
  - Let $n \in \mathbb{Z}^+$, for $x, y \in \mathbb{Z}$, the modulo n relation $\mathscr{R}$ is defined by $x\mathscr{R}y$, if $x - y$ is a multiple of n

Equivalence Ration

- A relation $\mathscr{R}$ on a set A is called equivalence relation if it is reflexive, symmetric, and transitive

- Ex 7.16: Are the following relations equivalence relations?

$$\mathscr{R}_1 = \{(1,1),(2,2),(3,3)\}$$
$$\mathscr{R}_2 = \{(1,1),(2,2),(2,3),(3,2),(3,3)\}$$
$$\mathscr{R}_3 = \{(1,1),(1,3),(2,3),(3,1),(3,3)\}$$

Outline

7.1 Rations Revisited: Properties of Relations

7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

7.3 Partial Orders: Hasse Diagrams

7.4 Equivalence Rations and Partitions

7.5 Finite State Machines: The Minimization Process

Composite Relation

- If $\mathscr{R}_1 \subseteq A \times B$ and $\mathscr{R}_2 \subseteq B \times C$ then the composite relation $\mathscr{R}_1 \circ \mathscr{R}_2$ is a relation from A to C defined by

$$\mathscr{R}_1 \circ \mathscr{R}_2 = \{(x, z) | x \in A, z \in C, \exists y \in B \text{ s.t. } (x, y) \in \mathscr{R}_1, (y, z) \in \mathscr{R}_2\}$$

- Ex 7.17: Let $A = \{1, 2, 3, 4\}, B = \{w, x, y, z\}, C = \{5, 6, 7\}$. If $\mathscr{R}_1 = \{(1, x), (2, x), (3, y), (3, z)\}$ and $\mathscr{R}_2 = \{(w, 5), (x, 6)\}$. Write $\mathscr{R}_1 \circ \mathscr{R}_2$. If $\mathscr{R}_3 = \{(w, 5), (w, 6)\}$, what is $\mathscr{R}_1 \circ \mathscr{R}_3$?

Association and Powers

- Let $\mathscr{R}_1 \subseteq A \times B, \mathscr{R}_2 \subseteq B \times C, \mathscr{R}_3 \subseteq C \times B$, we have $\mathscr{R}_1 \circ (\mathscr{R}_2 \circ \mathscr{R}_3) = (\mathscr{R}_1 \circ \mathscr{R}_2) \circ \mathscr{R}_3$
  - There is no ambiguity if we write $\mathscr{R}_1 \circ \mathscr{R}_2 \circ \mathscr{R}_3$

- Powers of $\mathscr{R}$ on A are recursively defined by: (i) $\mathscr{R}^1 = \mathscr{R}$ and (ii) $\mathscr{R}^{n+1} = \mathscr{R} \circ \mathscr{R}^n, \text{ where } n \in \mathbb{Z}^+$
- Ex 7.19: If $A = \{1, 2, 3, 4\}, \mathscr{R} = \{(1, 2), (1, 3), (2, 4), (3, 2)\}$, what are $\mathscr{R}^2, \mathscr{R}^3, \mathscr{R}^4$?

Zero-One Matrix

- An *m* by *n* zero-one matrix $E = (e_{ij})_{m \times n}$, is a rectangular array with *m* rows and *n* columns, where each e_{ij} denotes the entry in the *i*th row and *j*th column, which can be either *0* or *1*

- Ex 7.20: *E* is a 3 by 4 zero-one matrix

$$E = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Relation Matrices

- Ex 7.21: Write the following relations into relation matrices $A = \{1,2,3,4\}, B = \{w,x,y,z\}, C = \{5,6,7\}$

$$\mathscr{R}_1 = \{(1,x),(2,x),(3,y),(3,z)\}$$

$$\mathscr{R}_2 = \{(w,5),(x,6)\}$$

$$M(\mathscr{R}_1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad M(\mathscr{R}_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M(\mathscr{R}_1)M(\mathscr{R}_2) = ?$$

- Note that, a convention used here is 1 + 1 = 1, which is called boolean addition

Some Properties

- Let A be the set with n elements. $\mathscr{R}$ is a relation on A. If $M(\mathscr{R})$ is the relation matrix for $\mathscr{R}$ then
  - $M(\mathscr{R}) = \mathbf{0}$ iff $\mathscr{R} = \varnothing$
  - $M(\mathscr{R}) = \mathbf{1}$ iff $\mathscr{R} = A \times A$
  - $M(\mathscr{R}^m) = M(\mathscr{R})^m$, for $m \in \mathbb{Z}^+$

Precedes, Identify Matrix, Transpose

- Let E and F be two m by n $(0,1)$ matrices. We say E precedes, or is less than F, and we write $E \leqslant F$ if $e_{ij} \leqslant f_{ij}, \forall 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n$

- Identify Matrix:

$$I_n = (\delta_{ij})_{n \times n}, \text{ where } \delta_{ij} = 1 \text{ if } i = j, \delta_{ij} = 0, \text{ o.w.}$$

- Transpose:

$$A^{\text{tr}} : a^*_{ji} = a_{ij}$$

Relations in Matrices

- Given a relation $\mathscr{R}$ on A, where $|A| = n$. Let M be the relation matrix of $\mathscr{R}$
  - $\mathscr{R}$ is reflexive iff $I_n \leqslant M$
  - $\mathscr{R}$ is symmetric iff $M = M^{\mathrm{tr}}$
  - $\mathscr{R}$ is transitive iff $M^2 \leqslant M$
  - $\mathscr{R}$ is antisymmetic iff $M \cap M^{\mathrm{tr}} \leqslant I_n$
    - where $1 \cap 1 = 1, 1 \cap 0 = 0 \cap 1 = 0, 0 \cap 0 = 0$

Directed Graph

- Let V be a finite set. A directed graph (or digraph) G on V is made up the elements of V, called the vertices or nodes of G, and a subset E, of $V \times V$, that contains the directed edges, or arcs, of G. The set V is called the vertex set of G, and the set E is called the edge set. $G = (V,E)$ denotes the graph.
- If $(a, b) \in E$, then there is an edge from a to b. Vertex a is called the origin, and b is called terminus. We say b is adjacent from a and a is adjacent to b.
- If $a \neq b$ then $(a, b) \neq (b, a)$. An edge from a to a if called a loop.

Examples of Digraphs

- Are there isolated vertices?
- Undirected edges $\{a,b\}=\{b,a\}$

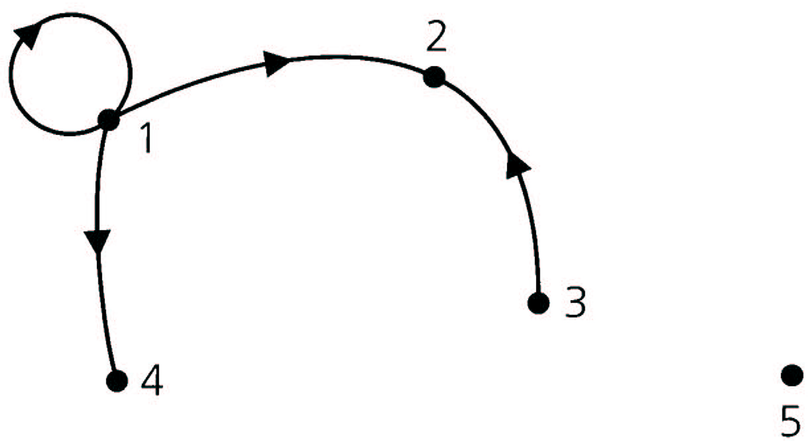


Figure 7.1

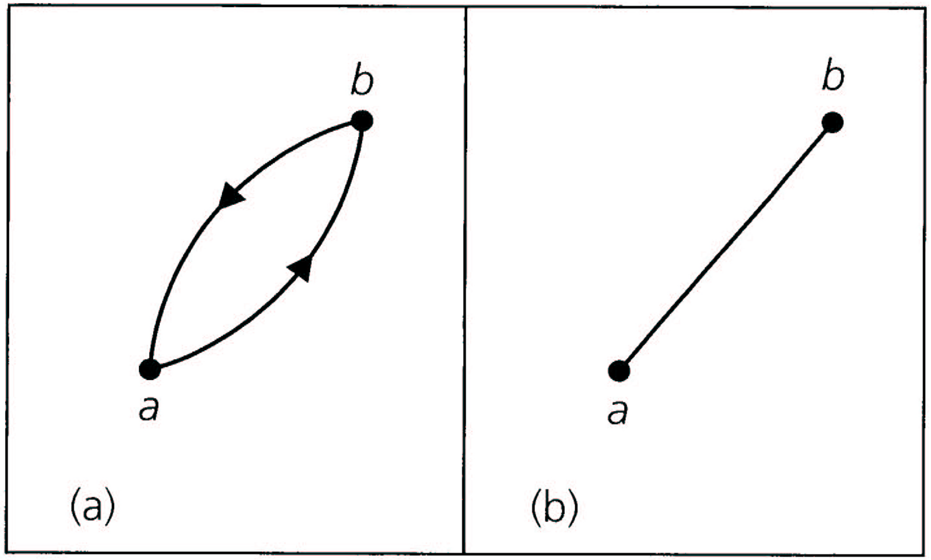


Figure 7.2

Precedence Graph

- Dependency among statements (computer programs)

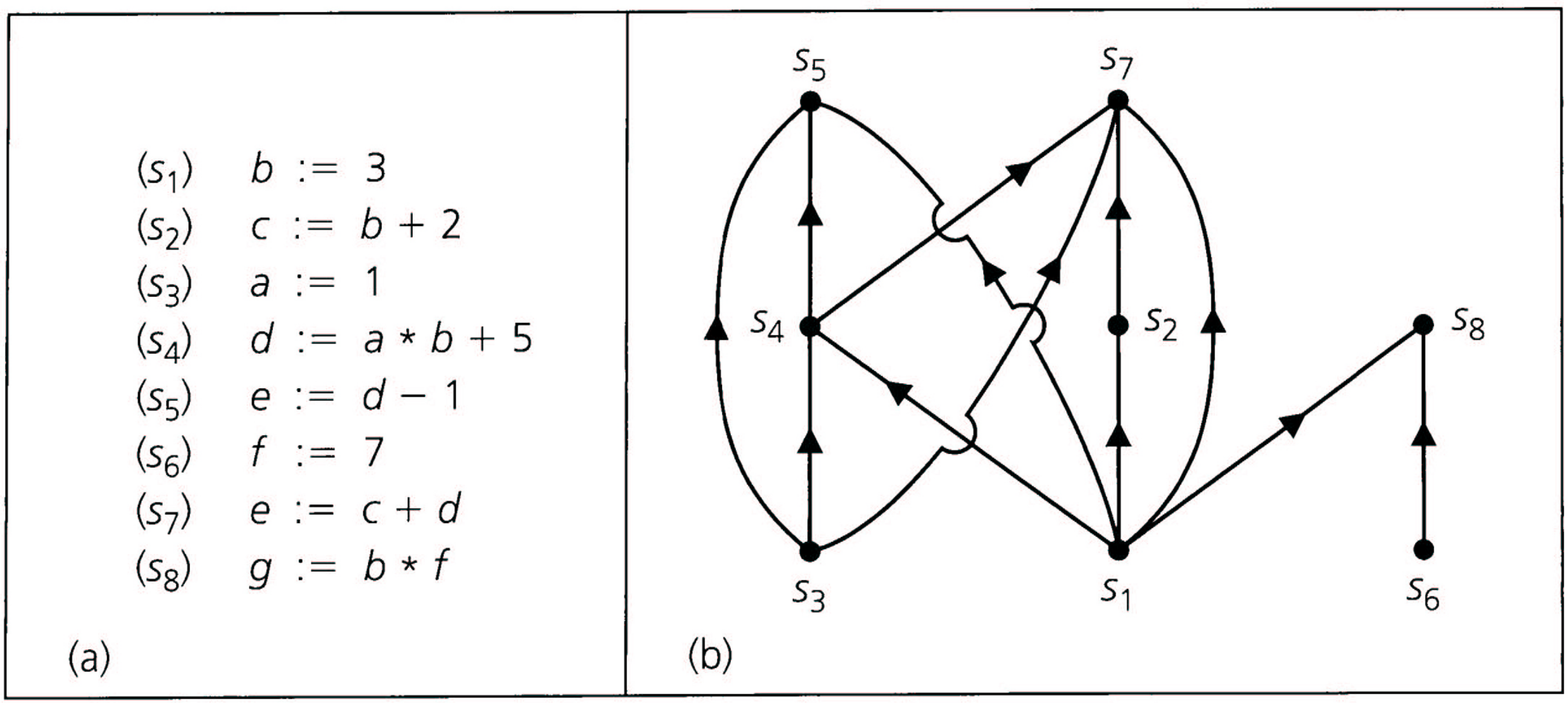


Figure 7.3

A Few More Terms

- What are: (i) associated undirected graph, (ii) path (cannot contain duplicated vertices), (iii) connected graph, (iv) length, (v) loop, and (vi) cycle?

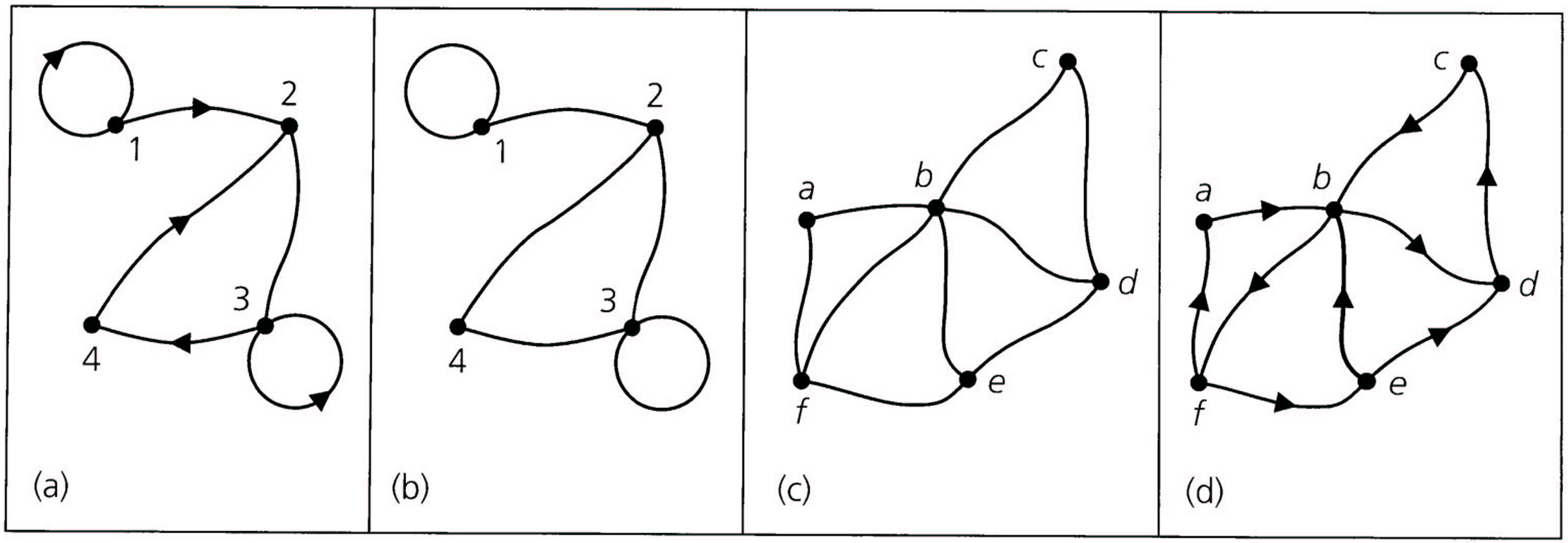
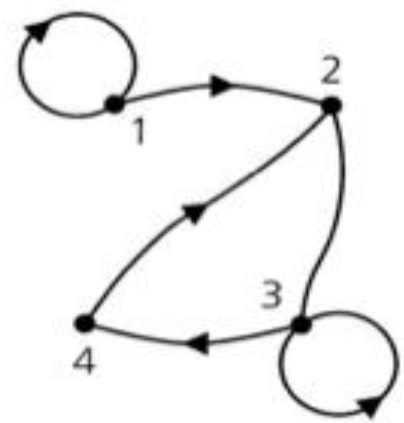


Figure 7.4

Strongly Connected

- A directed graph *G* on *V* is called strongly connected if there is a path from any vertex *x* to any vertex *y*



- The graph on the right is connected but not strongly connected
- The graph on the right is strongly connected and loop-free

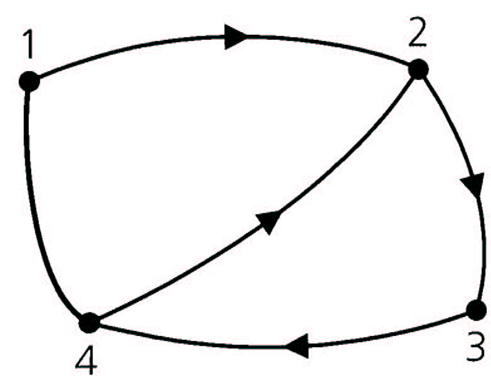


Figure 7.5

Components

- Two components in each graph

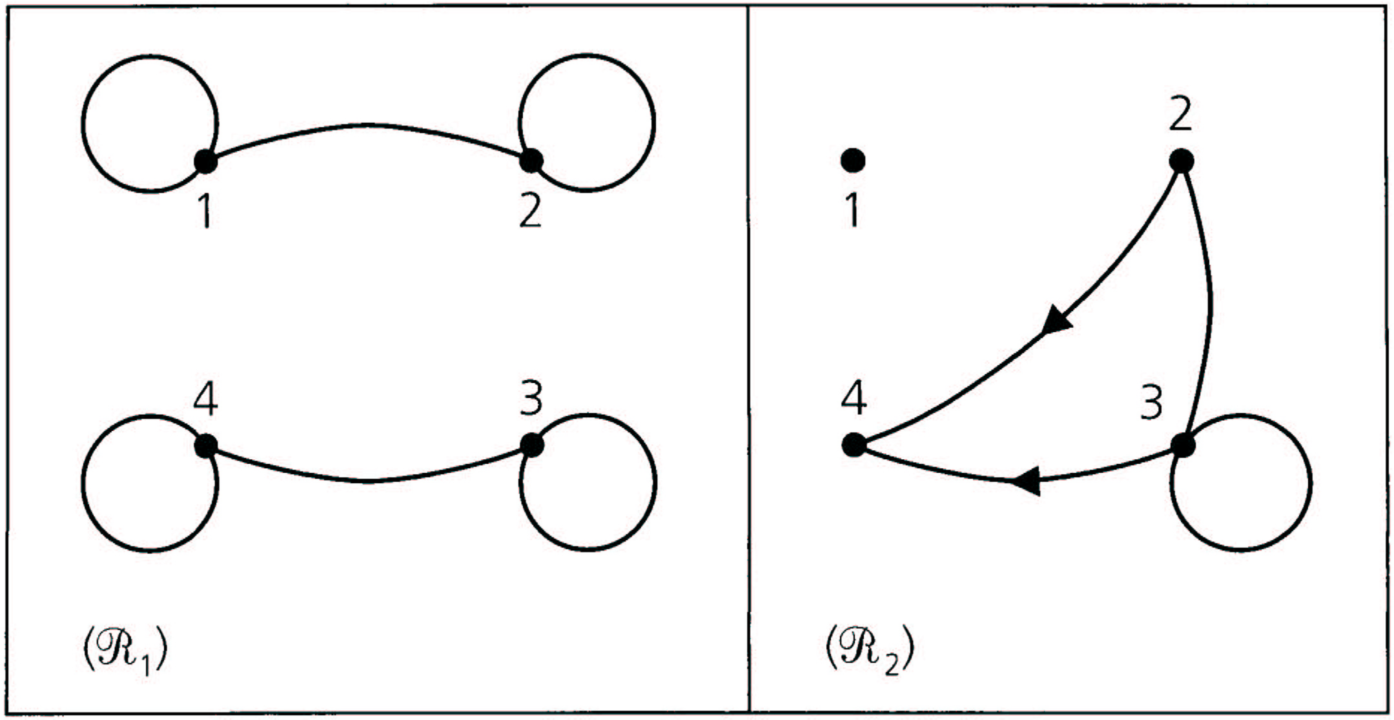


Figure 7.6

Complete Graphs

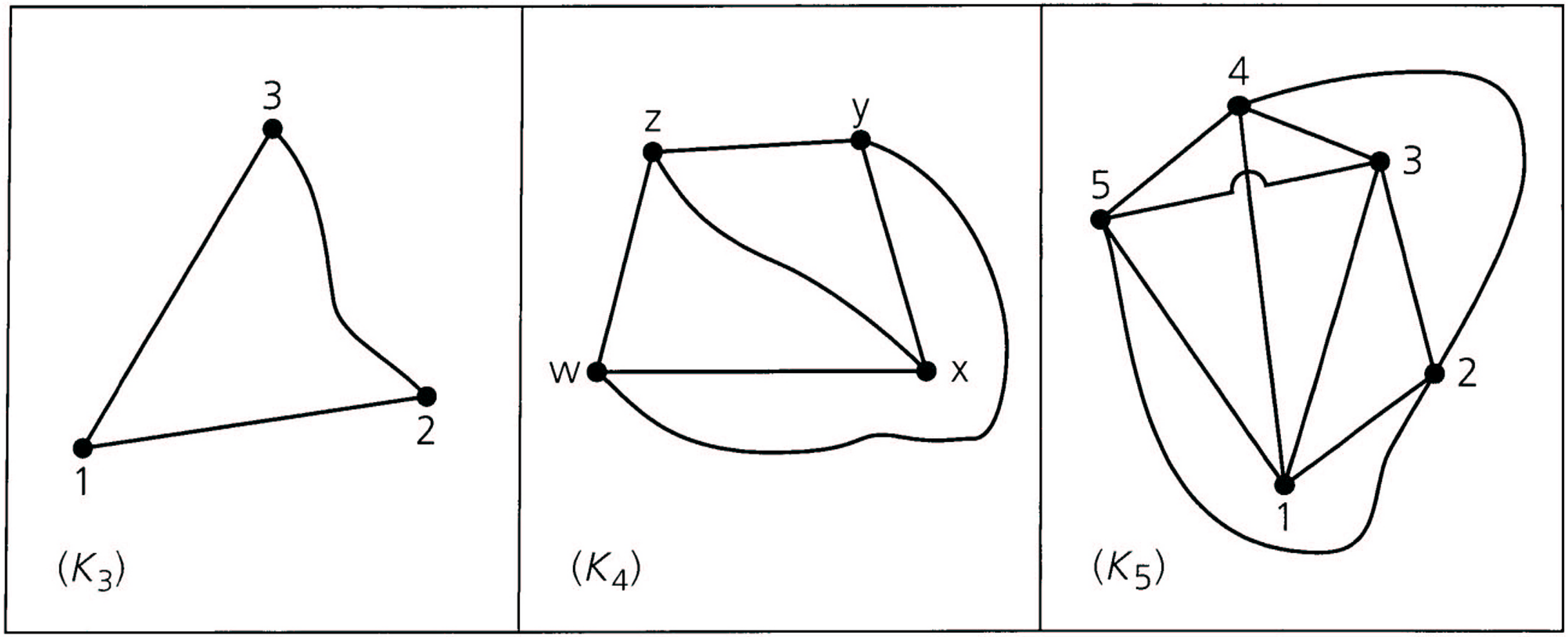


Figure 7.7

Matrices and Graphs

- A graph G describes a relation $\mathscr{R}$
  - If (x,y) is an edge in G, then $x\mathscr{R}y$

- Both 0-1 matrix and digraph can describe relations
  - The matrix is called the adjacency matrix for G
  - Or a relation matrix for $\mathscr{R}$

Reflexive and Antisymmetric

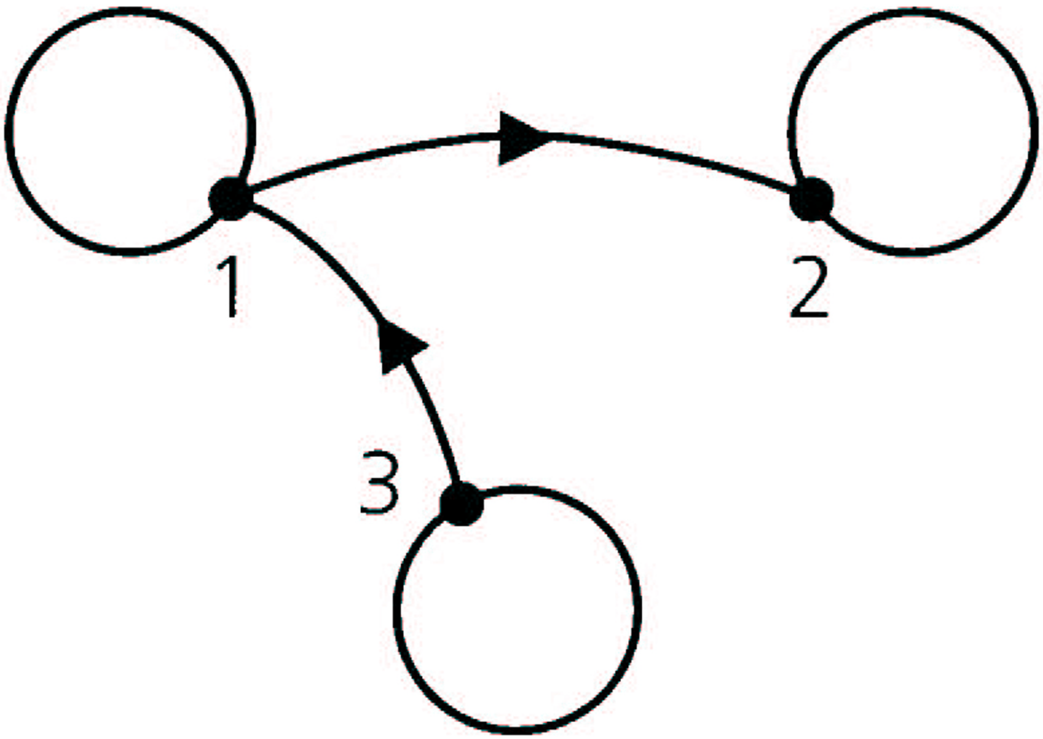


Figure 7.8

Symmetric

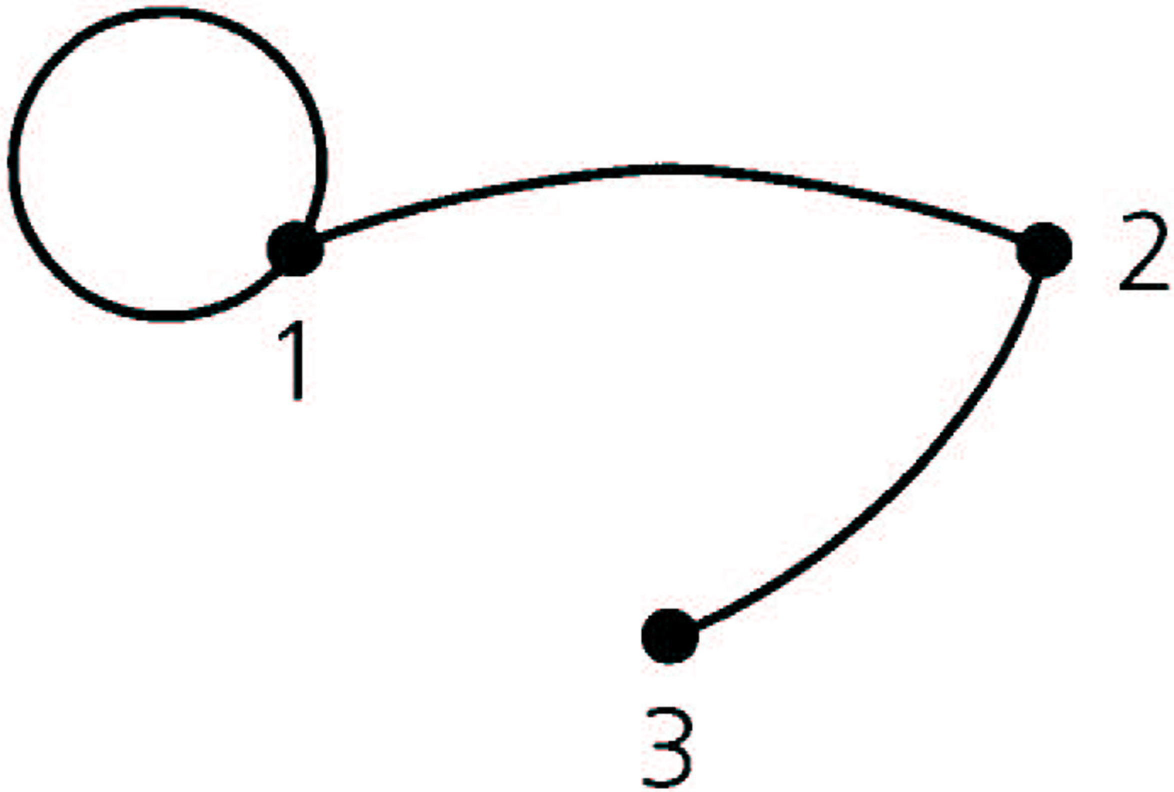


Figure 7.9

Transitive and Antisymmetric

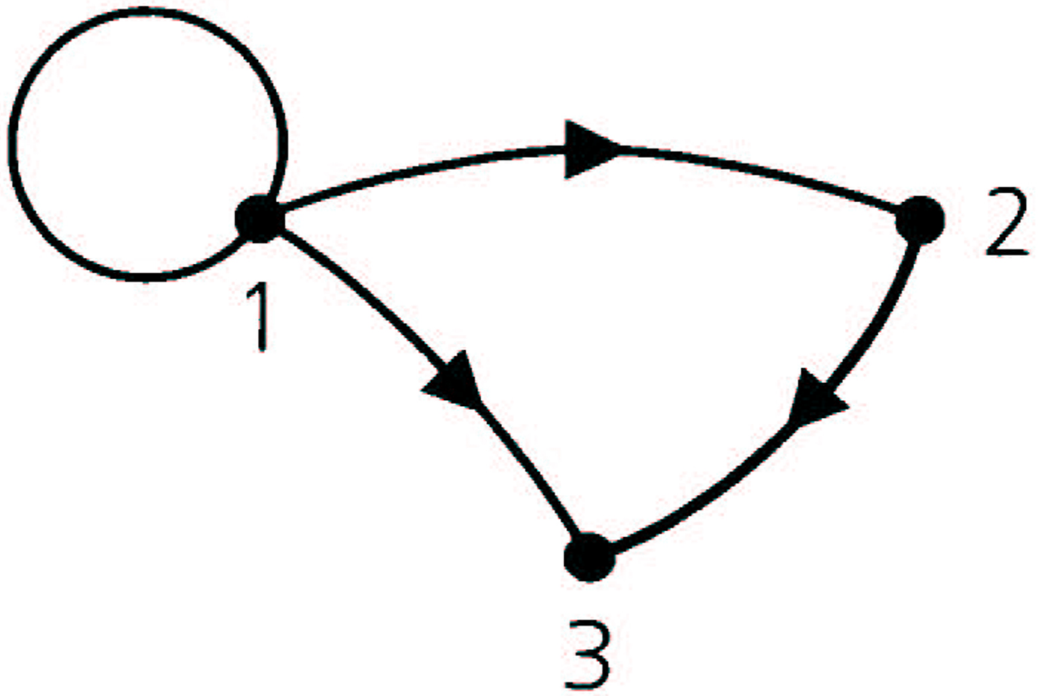


Figure 7.10

Outline

7.1 Rations Revisited: Properties of Relations

7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

7.3 Partial Orders: Hasse Diagrams

7.4 Equivalence Rations and Partitions

7.5 Finite State Machines: The Minimization Process

Partially Ordered Set

- $\mathscr{R}$ is a relation on A. $(A, \mathscr{R})$ is called partially ordered set if relation $\mathscr{R}$ on A is a partial order relation
  - Reflexive, antisymmetric, transitive
  - Also called poset

- Ex 7.34: Define the relation $x\mathscr{R}y$ if x, y are the same course or if x is a prerequisite of y

Not Partial Order

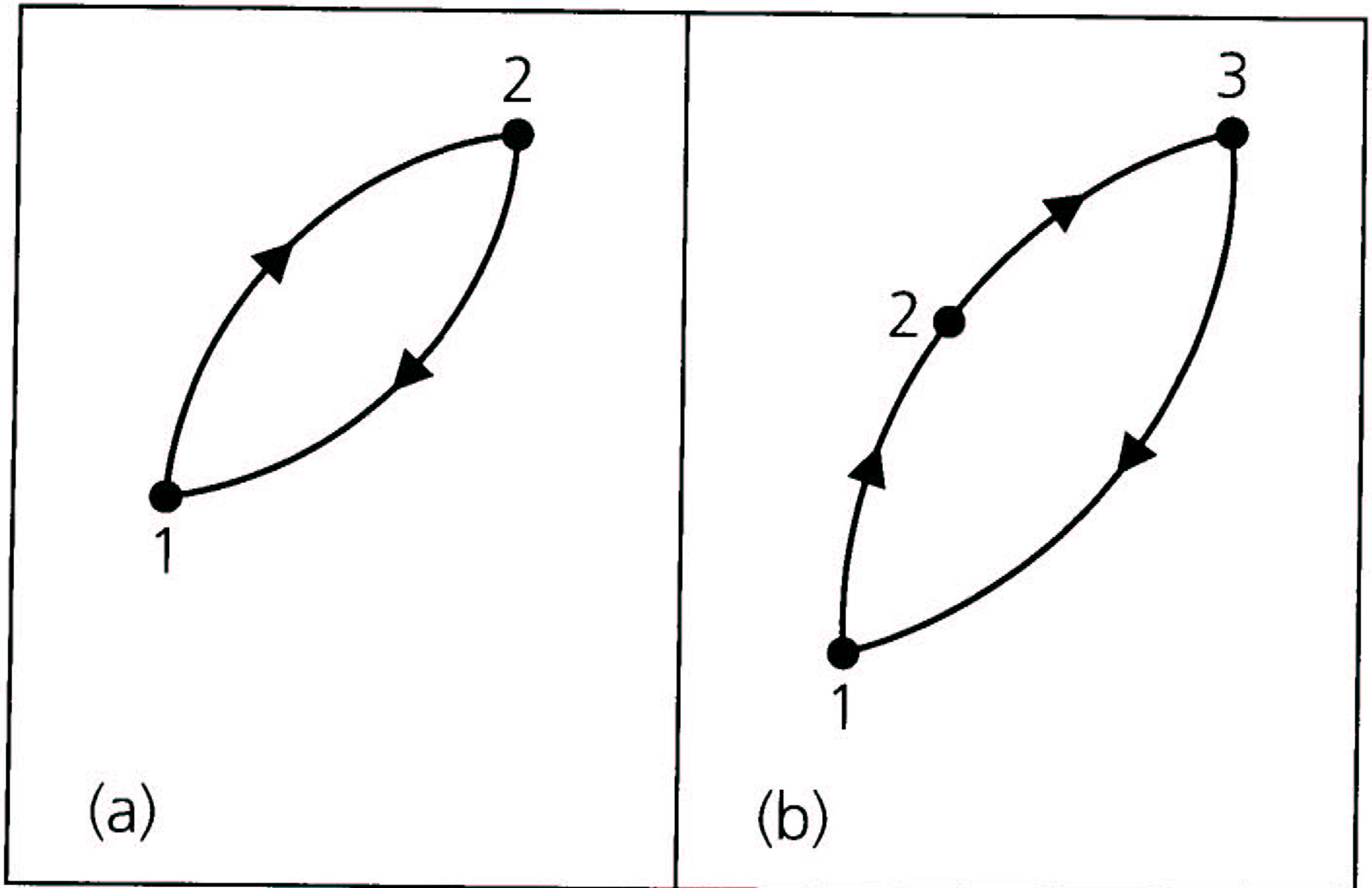


Figure 7.16

Hasse Diagram

- Drop loops
- Drop transitive edge
- Directions go from bottom up

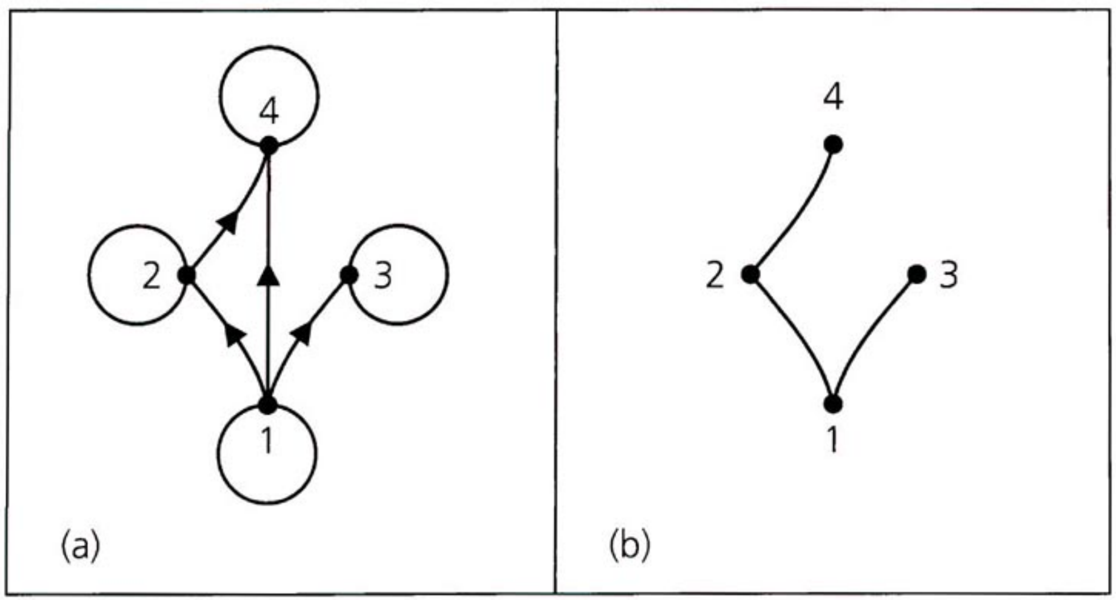


Figure 7.17

Totally Ordered

- If $(A, \mathscr{R})$ is a poset, A is totally order (or linearly ordered) if for any x and y, either $x\mathscr{R}y$ or $y\mathscr{R}x$.
  - $\mathscr{R}$ is called a total order (or a linear order)

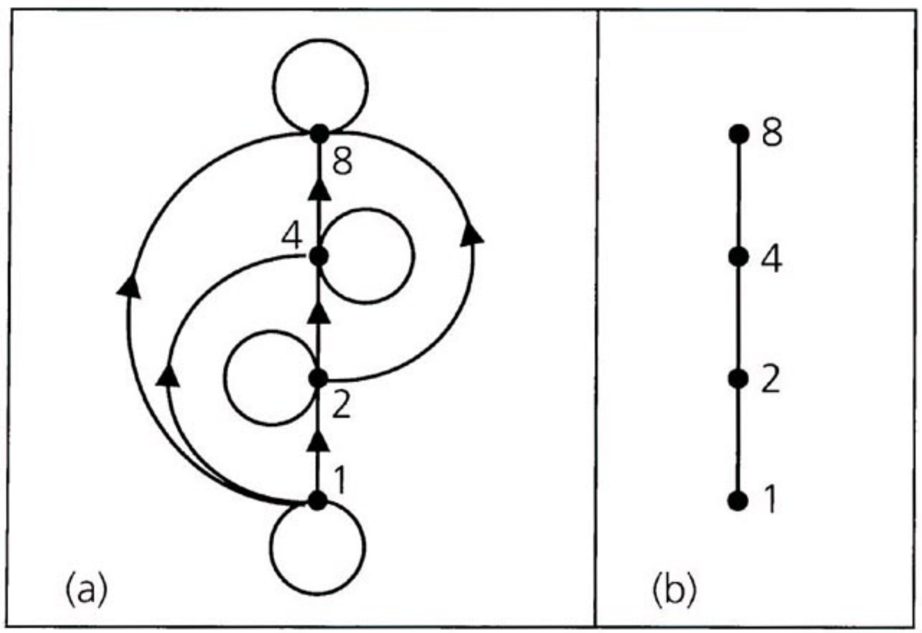


Figure 7.19

Partial vs. Total Orders

- Consider a car manufacturer which needs to assemble 7 components into a car. The partial order is $\mathscr{R}$ given below
  - Can the company find a total order $\mathscr{T}$ so that $\mathscr{R} \subseteq \mathscr{T}$?
  - Topological sorting!

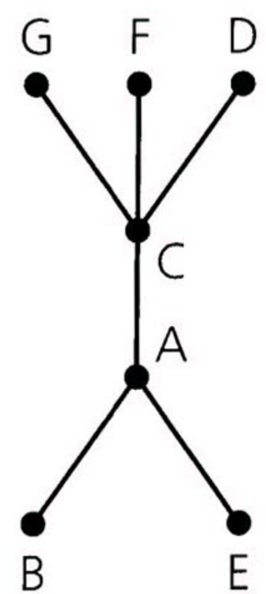


Figure 7.20

Topological Sorting

- Idea: Repeatedly remove the vertex that is not a source (nor an implicit source) of any edge, until we have no vertex left in the Hasse diagram

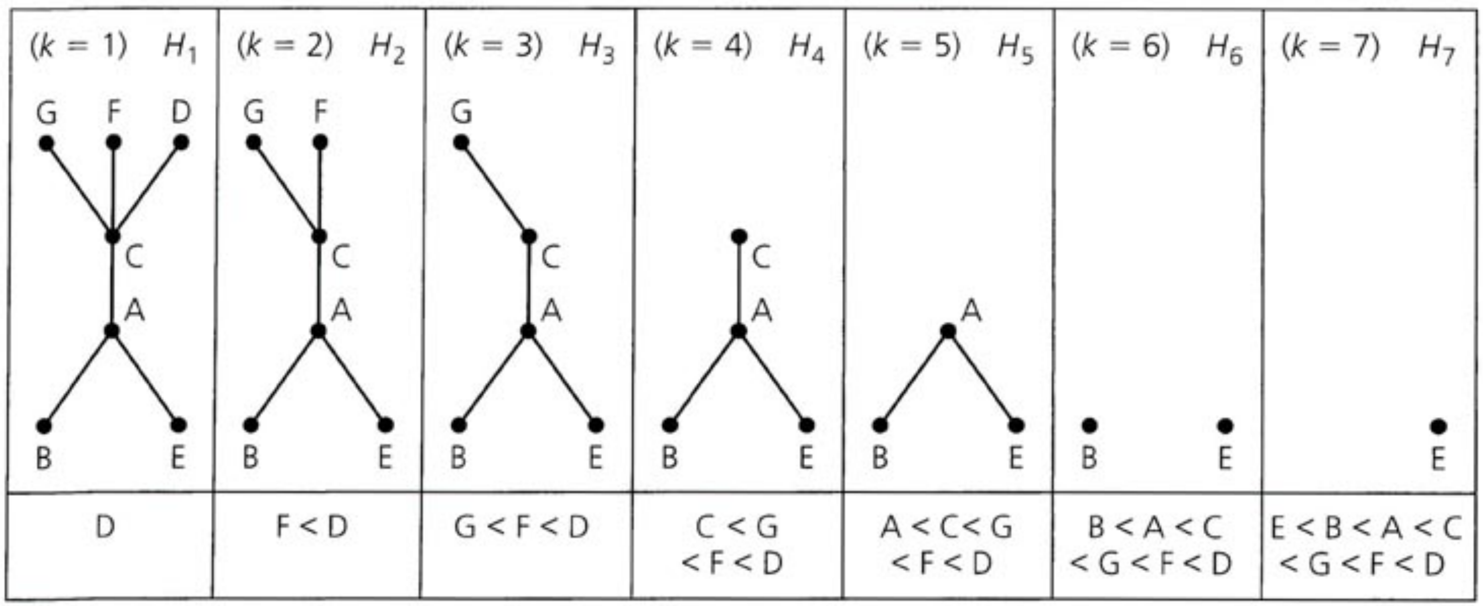


Figure 7.21

Topological Sorting Algorithm

- Input: A partial order $\mathscr{R}$ on a set A, where $|A| = n$
- Step 1: Let $k = 1$, Let H_1 be the Hasse diagram
- Step 2: Select v_k from H_k, so that no (implicitly directed) edge in H_k starts at v_k
- Step 3: If $k < n$, remove v_k and edges terminating at v_k from H_k. Call the new Hasse H_{k-1}, and goto step 1
- Step 4: The total order that contains $\mathscr{R}$ is

$$\mathscr{T} : v_n < v_{n-1} < \cdots < v_2 < v_1$$

Maximal, Minimal Elements

- If $(A, \mathscr{R})$ is a poset, an element $x \in A$ is a maximal element of A if for all $a \in A,\ a \neq x \Longrightarrow \neg(x\mathscr{R}a)$. An element $y \in A$ is a minimal element of A if for all $b \in A,\ b \neq y \Longrightarrow \neg(b\mathscr{R}y)$
- Ex 7.42: Define $\mathscr{R}$ be "less than or equal to" relation on $\mathbb{Z}$, we find that $(\mathbb{Z}, \mathscr{R})$ is a poset with no maximal nor minimal element. How about $(\mathbb{N}, \mathscr{R})$?
- A poset may have multiple maximal (minimal) elements! Recall the topological sorting algorithm.
- If $(A, \mathscr{R})$ is a poset and A is finite, A has both a maximal and a minimum element

Least, Greatest Elements

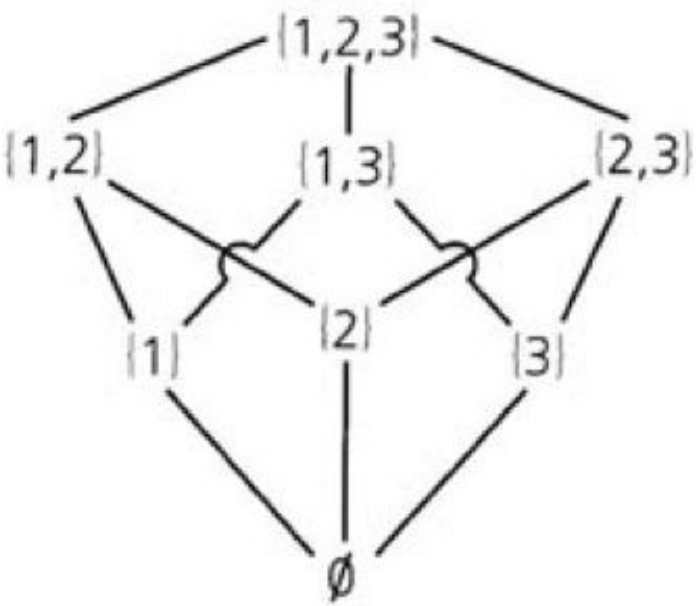
- If $(A, \mathscr{R})$ is a poset, an element $x \in A$ is a least element of A if $x \mathscr{R} a \ \forall a \in A$. An element $y \in A$ is a greatest element of A if $b \mathscr{R} y \ \forall b \in A$
  - If a poset has a greatest (least) element, the element is unique
- Ex 7.45: Define $\mathscr{U} = \{1, 2, 3\}$, $\mathscr{R}$ be subset relation
  - Poset $(\mathscr{P}(\mathscr{U}), \subseteq)$ has $\varnothing$ as a least element and $\mathscr{U}$ as a greatest element
  - Let A be all the nonempty subsets of $\mathscr{U}$. $(A, \subseteq)$ has $\mathscr{U}$ as the greatest element. It has no least element, but three minimal elements.

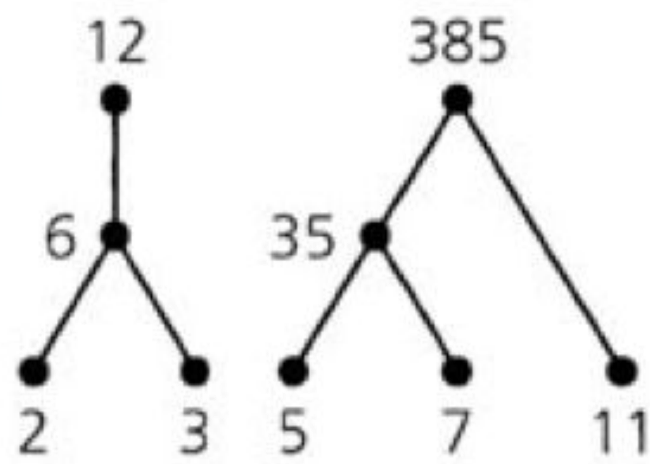
Lower and Upper Bounds

- If $(A, \mathscr{R})$ is a poset and $B \subseteq A$. An element $x \in A$ is called a lower bound of B if $x\mathscr{R}b \; \forall b \in B$. An element $y \in A$ is called an upper bound of B if $b\mathscr{R}y \; \forall b \in B$
  - $x' \in A$ is a greatest lower bound (glb) of B if it is a lower bound of B and $x''\mathscr{R}x'$ for any other lower bound x'' of B
  - $x' \in A$ is a least upper bound (lub) of B if it is an upper bound of B and $x'\mathscr{R}x''$ for any other upper bound x'' of B
- Ex 7.47: Let $A = \mathscr{P}(\{1, 2, 3, 4\})$ and $\mathscr{R}$ be the subset relation on A. If $B = \{\{1\}, \{2\}, \{1, 2\}\}$ then what are the upper bounds? What is the least upper bound? What is the greatest lower bound?
  - Lub and glb are unique

Lattice

- A poset $(A, \mathscr{R})$ is called a lattice if for all $x, y \in A$ the elements $\text{lub}\{x, y\}$ and $\text{glb}\{x, y\}$ both exist in A





Outline

Equivalence Relations

- A relation $\mathscr{R}$ on A is an equivalence relation if it's reflexive, symmetric, and transitive.

- Ex 1: For $A \neq \varnothing$, the equality relation is an equivalence relation, in which two elements are related if they are identical.

- Ex 2: Consider a relation on $\mathbb{Z}$, where $x\mathscr{R}y$ if $x - y$ is a multiple of 2.
  - How does this relation split $\mathbb{Z}$ into two subsets?

Partition

- Let A be a set and I be an index set, where A_i is not empty and $A_i \subseteq A$, for all $i \in I$. $\{A_i\}_{i\in I}$ is a partition of A if
  - $A = \bigcup_{i \in I} A_i$
  - $A_i \bigcap A_j = \varnothing$ for all $i \neq j;\ i, j \in I$

  Each subset A_i is a cell, or block of the partition
- Ex 7.52: For $A = \{1,2,3,\ldots,10\}$, the following are partitions of A
  - $\{\{1,2,3,4,5\}, \{6,7,8,9,10\}\}$
  - $A_i=\{i, i+5\}$, $1 \leqslant i \leqslant 5$

Equivalence Class

- Let $\mathscr{R}$ be an equivalence relation on A. The equivalence class of $x \in A$, denoted as $[x]$, is defined by $[x] = \{y | y \in A, y\mathscr{R}x\}$

- Ex 7.52: $\mathscr{R}$ is a equivalence relation on $\mathbb{Z}$, where $x\mathscr{R}y$ if $4|(x-y)$. The four equivalence classes are
  - $[0] = \{4k | k \in \mathbb{Z}\}$
  - $[1] = \{4k+1 | k \in \mathbb{Z}\}$
  - $[2] = \{4k+2 | k \in \mathbb{Z}\}$
  - $[3] = \{4k+3 | k \in \mathbb{Z}\}$

Properties of Equivalence Class

- Let $\mathscr{R}$ is an equivalence relation on A, and $x, y \in A$.
  - $x \in [x]$
  - $x \mathscr{R} y$ iff $[x] = [y]$
  - $[x] = [y]$ or $[x] \cap [y] = \varnothing$

- This theorem tells us the distinct equivalence classes given by $\mathscr{R}$ gives us a partition of A

Examples of Partitions

- Ex 7.56 (a) : Let $A = \{1, 2, 3, 4, 5\}$ and

$$\mathscr{R} = \{(1,1),(2,2),(2,3),(3,2),(3,3),(4,4),(4,5),(5,4),(5,5)\}$$

what's the corresponding partition?

- Ex 7.56 (b): Function $f : A \to B$, where $A = \{1, 2, 3, 4, 5, 6, 7\}$ and $B = \{x, y, z\}$, f is defined as

$$\{(1,x),(2,x),(3,x),(4,y),(5,z),(6,y),(7,x)\}$$

- We define a relation $\mathscr{R}$ by $a\mathscr{R}b$ if $f(a) = f(b)$. What is the partition determined by $\mathscr{R}$?

Examples of Partitions (cont.)

- If an equivalence relation $\mathscr{R}$ on $A = \{1, 2, 3, 4, 5, 6, 7\}$ results in the partition $A = \{1, 2\} \cup \{3\} \cup \{4, 5, 7\} \cup \{6\}$, what is $\mathscr{R}$? What's the size of it?

$$\mathscr{R} = (\{1, 2\} \times \{1, 2\}) \cup (\{3\} \times \{3\}) \cup (\{4, 5, 7\} \times \{4, 5, 7\}) \cup (\{6\} \times \{6\})$$

Equivalence Class and Partition

- For a set A
  - Any equivalence relation $\mathscr{R}$ on A leads to a partition of A
  - Any partition of A gives an equivalence relation $\mathscr{R}$ on A

- For any set A, there is a one-to-one correspondence between the set of equivalence relations on A and the set of partitions of A.
  - So counting the number of partitions is the same as counting 1-1 functions.

Outline

Redundant States

- Redundant state: A state that can be eliminated because other states will perform its function
- Consider a finite state machine $M = (S, \mathscr{I}, \mathscr{O}, \nu, \omega)$, Let a relation $s_1 E_1 s_2$ if $\omega(s_1, x) = \omega(s_2, x)$ for all $x \in \mathscr{I}$
  - E_1 is called *1*-equivalent.
- $s_1 E_k s_2$ if $\omega(s_1, x) = \omega(s_2, x)$ for all $x \in \mathscr{I}^k$
  - E_k is called *k*-equivalent
- $s_1 E s_2$ if $s_1 E_k s_2$ is true for all $k \geqslant 1$
  - E is called equivalent

Minimization Algorithm

- To get rid of redundant states
- Step 1: Let $k=1$, find states that are 1-equivalent by examining the output rows in the state table. This gives partition P_1 and relation E_1
- Step 2: When P_k is found, we obtain P_{k+1} by knowing that if $s_1 E_k s_2$, then $s_1 E_{k+1} s_2$ when $\nu(s_1, x) E_k \nu(s_2, x)\ \forall x \in \mathscr{I}$
  - This is true if $\nu(s_1, x)$ and $\nu(s_2, x)$ are in the same cell of P_k
- Step 3: If $P_{k+1}=P_k$, we are done, o.w. goto step 2

A Simple Example

- Ex 7.60: If $\mathscr{I} = \mathscr{O} = \{0, 1\}$, the state table is given below. What is P_1? $P_1 : \{s_1\}, \{s_2, s_5, s_6\}, \{s_3, s_4\}$
- Show $\nu(s_3, x) E_1 \nu(s_4, x)$, and thus?
- Show $\neg[\nu(s_5, x) E_1 \nu(s_6, x)]$, and thus?
- $P_2 : \{s_1\}, \{s_2, s_5\}, \{s_6\}, \{s_3, s_4\}$
- Since $P_1 \neq P_2$, we need to get P_3
  - Because $P_3 = P_2$, we stop here
  - s_5, s_4 are redundant states

Table 7.1

	ν		ω	
	0	1	0	1
s_1	s_4	s_3	0	1
s_2	s_5	s_2	1	0
s_3	s_2	s_4	0	0
s_4	s_5	s_3	0	0
s_5	s_2	s_5	1	0
s_6	s_1	s_6	1	0

Refinement

- P_2 is called a refinement of P_1, $P_2 \leqslant P_1$, if every cell of P_2 is contained in a cell of P_1. When $P_2 \leqslant P_1$ and $P_2 \neq P_1$, we write $P_2 = P_1$.

- In the minimization process, if $k>=1$ and $P_k=P_{k+1}$, then $P_{r+1}=P_r$ for all $r >=k+1$

Distinguishing String

- A sample string with length k+1that leads to different outputs for states s_1 and s_2

- Ex 7.61: Find the minimal distinguish string for s_2 and s_6 in the finite state machine of Ex 7.60

$$P_2 : \{s_1\}, \{s_2, s_5\}, \{s_6\}, \{s_3, s_4\}$$

$$P_1 : \{s_1\}, \{s_2, s_5, s_6\}, \{s_3, s_4\}$$

Table 7.1

	ν		ω	
	0	1	0	1
s_1	s_4	s_3	0	1
s_2	s_5	s_2	1	0
s_3	s_2	s_4	0	0
s_4	s_5	s_3	0	0
s_5	s_2	s_5	1	0
s_6	s_1	s_6	1	0