

# **CS 2336: Discrete Mathematics**

## **Chapter 8**

### **The Principle of Inclusion and Exclusion**

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# Outline

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**8.1 The Principle of Inclusion and Exclusion**

**8.2 Generalizations of the Principle**

**8.3 Derangement: Nothing is in Its Right Place**

**8.4 Rook Polynomials**

**8.5 Arrangements with Forbidden Positions**

# Notations

- Let  $S$  be a set and  $|S|=N$ . Let  $c_1, c_2, \dots, c_t$  be a collection of  $t$  **conditions** or **properties**, each may be satisfied by some elements of  $S$ .
- For an  $1 \leq i \leq t$ ,  $N(c_i)$  denotes the number of elements in  $S$  that satisfy condition  $c_i$ .
- $N(c_i c_j)$  denotes the number of elements in  $S$  that satisfy both conditions  $c_i$  and  $c_j$ , and perhaps others.
- For an  $1 \leq i \leq t$ ,  $N(\bar{c}_i) = N - N(c_i)$  denotes the number of elements not satisfy condition  $c_i$ .
- Also define  $N(\bar{c}_i \bar{c}_j)$  and  $N(\overline{c_i c_j})$  .

# Principle of Inclusion and Exclusion

- Number of elements of  $S$  that satisfy none of the condition  $c_i, 1 \leq i \leq t$ , is denoted by  $\bar{N} = N(\bar{c}_1 \bar{c}_2 \cdots \bar{c}_t)$

$$\begin{aligned} \bar{N} = & N - [N(c_1) + N(c_2) + \cdots + N(c_t)] \\ & + [N(c_1 c_2) + N(c_1 c_3) + \cdots + N(c_1 c_t) + N(c_2 c_3) + \cdots + N(c_{t-1} c_t)] \\ & - [N(c_2 c_2 c_3) + N(c_1 c_2 c_4) + \cdots + N(c_1 c_2 c_t) + N(c_1 c_3 c_4) + \cdots \\ & + N(c_1 c_3 c_t) + \cdots + N(c_{t-2} c_{t-1} c_t)] \\ & + \cdots + (-1)^t N(c_1 c_2 c_3 \cdots c_t) \end{aligned}$$

or

$$\begin{aligned} \bar{N} = & N - \sum_{1 \leq i \leq t} N(c_i) + \sum_{1 \leq i, j \leq t} N(c_i c_j) - \sum_{1 \leq i, j, k \leq t} N(c_i c_j c_k) + \cdots \\ & + (-1)^t N(c_1 c_2 c_3 \cdots c_t) \end{aligned}$$

# Proof by Counting

$$\bar{N} = N - \sum_{1 \leq i \leq t} N(c_i) + \sum_{1 \leq i, j \leq t} N(c_i c_j) - \sum_{1 \leq i, j, k \leq t} N(c_i c_j c_k) + \dots$$
$$+ (-1)^t N(c_1 c_2 c_3 \dots c_t)$$

- Goal: show any  $x \in S$  contributes the same amount (either 0 or 1) to LHS and RHS
- Case I: What if  $x$  satisfies none of the conditions?
- Case II: If  $x$  satisfies **exactly**  $r$  conditions. LHS is always 0. RHS is
  - (1) once in  $N$  (first term)
  - (2)  $r$  times in the second terms

# Proof by Counting (cont.)

$$\bar{N} = N - \sum_{1 \leq i \leq t} N(c_i) + \sum_{1 \leq i, j \leq t} N(c_i c_j) - \sum_{1 \leq i, j, k \leq t} N(c_i c_j c_k) + \cdots \\ + (-1)^t N(c_1 c_2 c_3 \cdots c_t)$$

- Case II: If  $x$  satisfies **exactly**  $r$  conditions. RHS is

- (3)  $\binom{r}{2}$  times in the third terms
- (4)  $\binom{r}{3}$  times in the fourth terms
- .....
- (Last)  $\binom{r}{r}$  time in the last term.

- Hence, the RHS is

$$1 - r + \binom{r}{2} - \binom{r}{3} + \cdots + (-1)^r \binom{r}{r} = [1 + (-1)]^r = 0^r = 0$$

- Per Corollary 1.1

# More Notations

- The number of elements in  $S$  that satisfy at least one condition  $c_i$ , is given by  $N(c_1 \text{ or } c_2 \text{ or } \cdots \text{ or } c_t) = N - \bar{N}$

- For brevity, we write

$$S_0 = N$$

$$S_1 = [N(c_1) + N(c_2) + \cdots + N(c_t)]$$

.....

$$S_k = \sum N(c_{i_1}, c_{i_2}, \cdots, c_{i_k}), \quad 1 \leq k \leq t$$

collection from  $t$  conditions, hence it has  $\binom{t}{k}$  entries

- Hence we have  $\bar{N} = S_0 - S_1 + S_2 - S_3 + \cdots + (-1)^t S_t$

# Simple Example 1

- Ex 8.4: Find the number of positive integer between 1 and 100 (inclusive), where  $n$  is not divisible by 2, 3, or 5
  - $c_1$ : if  $n$  is divisible by 2
  - $c_2$ : if  $n$  is divisible by 3
  - $c_3$ : if  $n$  is divisible by 5
- What is  $N(\bar{c}_1\bar{c}_2\bar{c}_3)$  ?

$$N(c_1) = \lfloor 100/2 \rfloor$$

$$N(c_1c_2) = \lfloor 100/(2 \times 3) \rfloor$$

.....



# Simple Example 2

- Ex 8.7: In how many ways can 26 letters be permuted so that none of the patterns: **car**, **dog**, **pun**, or **byte** occurs?
  - Let  $S$  be the set of all permutations of letters,  $|S| = 26!$
  - $c_i$  is the number of permutation contains the  $i$ -th pattern
- $N(c_1) = 24!$ ,  $N(c_2) = N(c_3) = 24!$ ,  $N(c_4) = 23!$
- $N(c_1c_3) = N(c_2c_3) = 22!$   $N(c_1c_4) = N(c_2c_4) = N(c_3c_4) = 21!$
- .....
- $N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4) = 26! - [3(24!) + 23!] + [3(22!) + 3(21!)] - [20! + 3(19!)] + 17!$

# Simple Example 3

- An engineer is building two-way roads to connect five villages, s.t. no village will be isolated. In how many ways can he do this? No-loop is considered.
- Let  $S$  be all loop-free undirected graphs  $G$  on  $V = \{a, b, c, d, e\}$ .  $S_0 = 2^{10}$ , because there are 10 possible two-way roads for five villages.

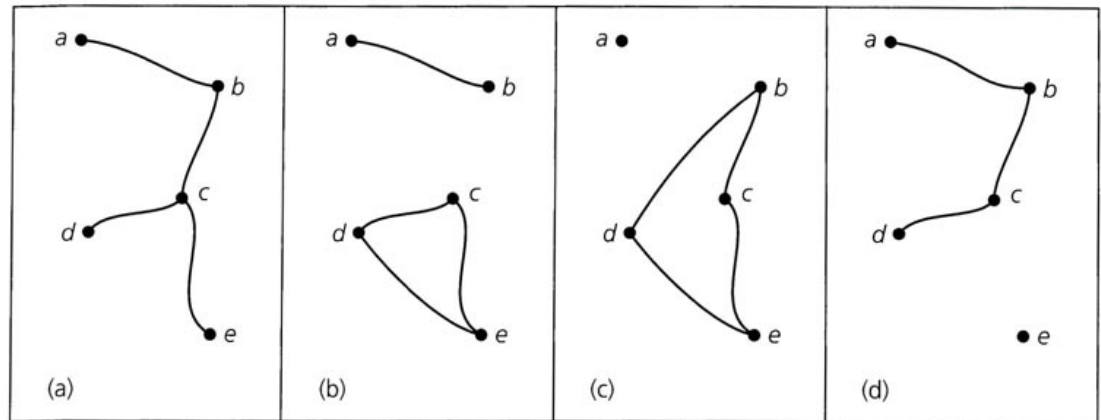


Figure 8.3

# Simple Example 3 (cont.)

- Let  $c_i$  be the condition that the  $i$ -th village is isolated. The answer to our problem is  $N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4\bar{c}_5)$
- $N(c_i)=2^6$ , why?
- $N(c_i c_j)=2^3$ , why?
- Ans:  $2^{10} - \binom{5}{1}2^6 + \binom{5}{2}2^3 - \binom{5}{3}2^1 + \binom{5}{4}2^0 - \binom{5}{5}2^0 = 768$

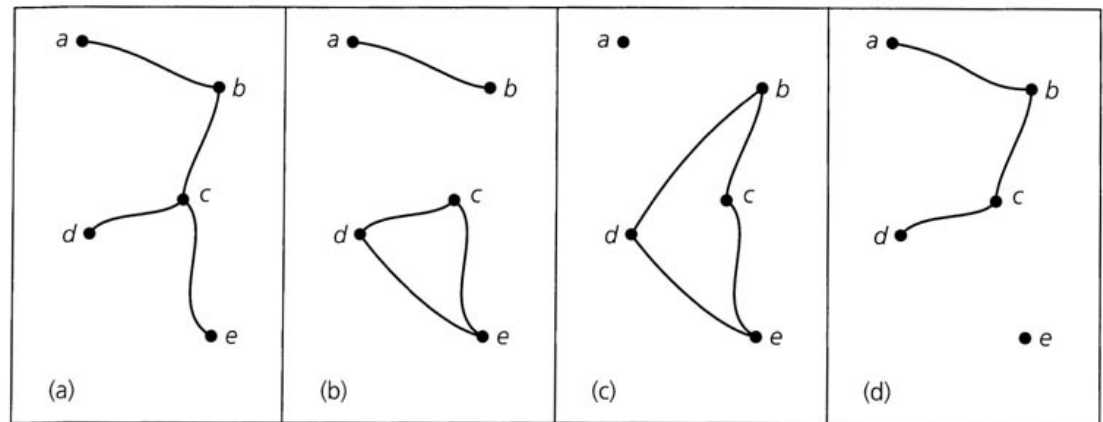


Figure 8.3

# Outline

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**8.1 The Principle of Inclusion and Exclusion**

**8.2 Generalizations of the Principle**

**8.3 Derangement: Nothing is in Its Right Place**

**8.4 Rook Polynomials**

**8.5 Arrangements with Forbidden Positions**

# Notations

- Let  $S$  be a set and  $|S|=N$ . Let  $c_1, c_2, \dots, c_t$  be a collection of  $t$  **conditions** or **properties**, each may be satisfied by some elements of  $S$ .
- Let  $E_m$  be the number of elements in  $S$  that satisfy exactly  $m$  of the  $t$  conditions
  - We knew how to compute  $E_0$
- $E_1 = N(c_1 \bar{c}_2 \cdots \bar{c}_t) + N(\bar{c}_1 c_2 \cdots \bar{c}_t) + \cdots + N(\bar{c}_1 \bar{c}_2 \cdots c_t)$
- $E_2 = N(c_1 c_2 \bar{c}_3 \cdots \bar{c}_t) + N(c_1 \bar{c}_2 c_3 \cdots \bar{c}_t) + \cdots + N(\bar{c}_1 \bar{c}_2 \bar{c}_3 \cdots c_{t-1} c_t)$

# Generalized Formula

- $E_m$  is given by:

$$E_m = S_m - \binom{m+1}{1} S_{m+1} + \binom{m+2}{2} S_{m+2} - \cdots + (-1)^{t-m} \binom{t}{t-m} S_t$$

- Proof Sketch:
- Case I:  $x$  satisfies few than  $m$  conditions. Contributes 0 to both LHS and RHS.
- Case II:  $x$  satisfies exactly  $m$  conditions. It is counted once in  $E_m$ , and once in  $S_m$ , but not in  $S_{m+1}, \dots, S_i$ .

# Generalized Formula (cont.)

$$E_m = S_m - \binom{m+1}{1} S_{m+1} + \binom{m+2}{2} S_{m+2} - \dots + (-1)^{t-m} \binom{t}{t-m} S_t$$

- Case III:  $x$  satisfies  $r$  conditions, where  $m < r \leq t$ .  $x$  contributes nothing in LHS. It is counted  $\binom{r}{m}$  times in  $S_m$ ,  $\binom{r}{m+1}$  times in  $S_{m+1}$ , ..., and  $\binom{r}{r}$  in  $S_r$ , but 0 time for anything beyond  $r$ . Hence we have the count:

$$\binom{r}{m} - \binom{M=1}{1} \binom{r}{m+1} + \binom{m+2}{2} \binom{r}{M=2} - \dots + (-1)^{r-m} \binom{r}{r-m} \binom{r}{r}$$

- some derivation leads to 0 in RHS.

# A Simple Example

- $E_1 = N(c_1) + N(c_2) + N(c_3) - 2[N(c_1c_2) + N(c_1c_3) + N(c_2c_3)] + 3N(c_1c_2c_3)$
- $E_2 = N(c_1c_2) + N(c_1c_3) + N(c_2c_3) - 3N(c_1c_2c_3)$
- $E_3 = N(c_1c_2c_3)$

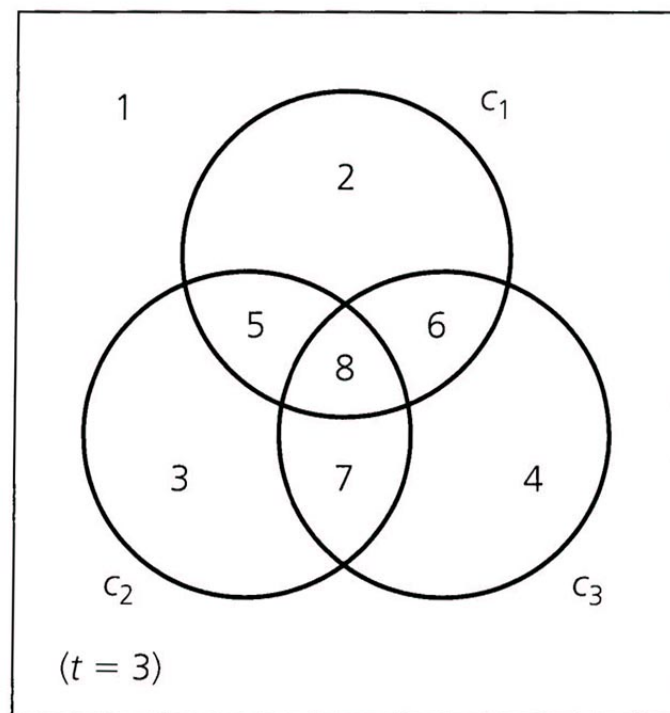


Figure 8.4



# Another Generalization

- Let  $L_m$  denotes the number of elements of  $S$  that satisfy **at least**  $m$  of the  $t$  conditions.

$$L_m = S_m - \binom{m}{m-1} S_{m+1} + \binom{m+1}{m-1} S_{m+2} - \cdots + (-1)^{t-m} \binom{t-1}{m-1} S_t$$

- What is  $L_2$  in the figure?

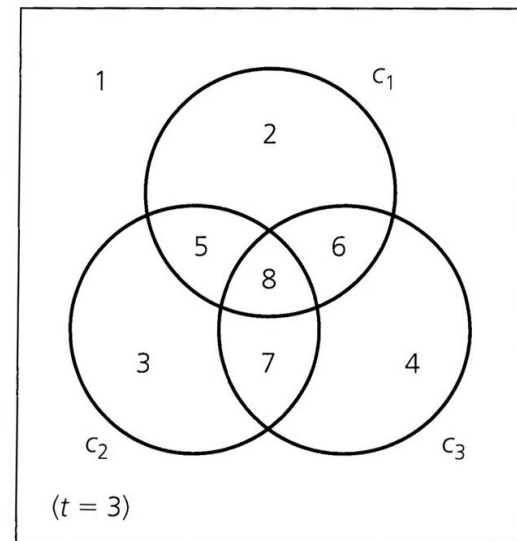


Figure 8.4

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# Maclaurin Series for Exp Function

- We know  $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
- Hence,  $e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \dots$
- With  $n \geq 7$ ,  $\sum_{n=0}^k \frac{(-1)^n}{n!}$  can approximate  $e^{-1}$  well

# Derangements

- Ex 8.12: Ralph bets on ten chosen horses in a race. In how many ways can they reach the finish line so that he loses all the bets.
- Equivalent to: In how many ways we can arrange  $1, 2, \dots, 10$  so that  $1$  is not in first place,  $2$  is not in second place,  $\dots$ , and  $10$  is not in tenth place?
- These arrangements are called **derangements** of  $1, 2, 3, \dots, 10$

# Derangements (cont.)

- An arrangement is said to satisfy condition  $c_i$ , if integer  $i$  is in the  $i$ -th place
- The number of derangements  $d_{10}$  is given by

$$\begin{aligned}d_{10} = N(\bar{c}_1 \bar{c}_2 \cdots \bar{c}_{10}) &= 10! - \binom{10}{1} 9! + \binom{10}{2} 8! - \binom{10}{3} 7! + \cdots + \binom{10}{10} 0! \\ &= 10! \left[ 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{1}{10!} \right] \approx (10!) e^{-1}\end{aligned}$$

- The probability that Ralph will lose every bet is about  $\frac{10! e^{-1}}{10!} = e^{-1}$ 
  - Not a bad approximation with number of horses is 11, 12, ..., 13

# Two Simple Examples

- Ex 8.13: The number of derangements of  $1, 2, 3, 4$   
$$d_4 = N(\bar{c}_1 \bar{c}_2 \bar{c}_3 \bar{c}_4) = 4! - \binom{4}{1} 3! + \binom{4}{2} 2! - \binom{4}{3} 1! + \binom{4}{4} 0!$$
$$= 4! \left( 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \right) = 12 - 4 + 1 = 9$$
- Ex 8.14: Peggy assigns 7 books to 7 reviewers: one book for each reviewer in the first week, and another book for each reviewer in the second week. How many ways can she distribute the books so that she gets two reviewers of each book?
  - $7!$  for the first week, so  $7!d_7 \approx 7!^2 e^{-1}$  in total

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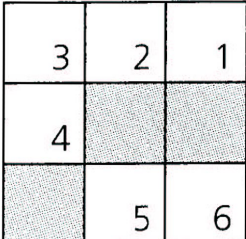
**8.3 Derangement: Nothing is in Its Right Place**

**8.4 Rook Polynomials**

**8.5 Arrangements with Forbidden Positions**

# Chessboard and Rook

- Shaded squares are not part of a chessboard  $C$
- Rook (or castle) can move horizontally or vertically over all unoccupied spaces in each turn
  - Ex: Where can a rook move if it's at 3, or 5?
- $R_k(C)$ : the number of ways,  $k$  rooks can be placed on the unshaded squares so that no two of them can take each other.



3	2	1
4		
	5	6

Figure 8.6



# Rook Polynomial

- $r_k(C)$ : the number of ways,  $k$  rooks can be placed on the unshaded squares so that no two of them can take each other.
- Rook Polynomial:  $r(C, x) = \sum_{\forall k} r_k(C) x^k$
- Ex:  $r_0(C)=1, r_1(C)=6, r_2(C)=8, r_3(C)=2$ 
  - Thus  $r(C, x) = 1 + 6x + 8x^2 + 2x^3$

	3	2	1
4			
		5	6

**Figure 8.6**

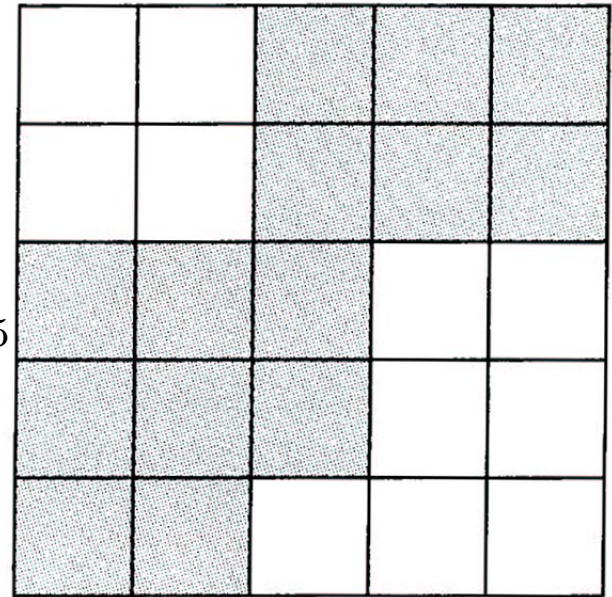
# Subboards

- The above approach is tedious, and we want to break a chessboard into multiple **subboards**
- 2 subboards:  $C_1$  (upper-left) and  $C_2$  (lower-right)
- They are **disjoint**, and we have

$$r(C_1, x) = 1 + 4x + 2x^2$$

$$r(C_2, x) = 1 + 7x + 10x^2 + 2x^3$$

$$r(C, x) = 1 + 11x + 40x^2 + 56x^3 + 28x^4 + 4x^5$$



**Figure 8.7**

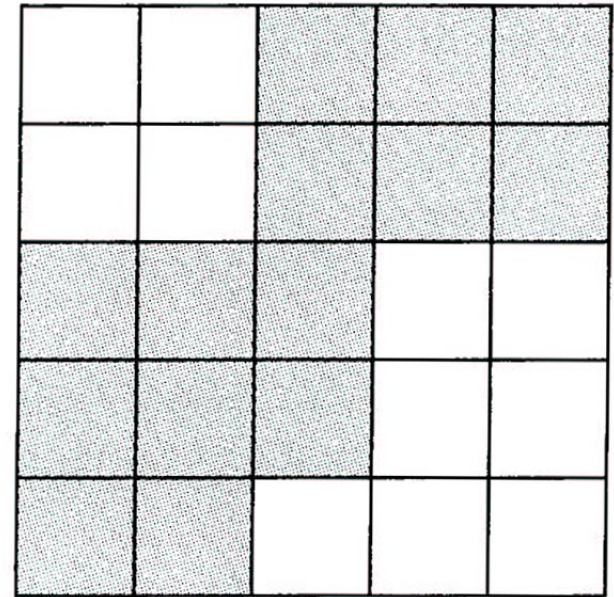
# Subboards (cont.)

$$r(C_1, x) = 1 + 4x + 2x^2$$

$$r(C_2, x) = 1 + 7x + 10x^2 + 2x^3$$

$$r(C, x) = 1 + 11x + 40x^2 + 56x^3 + 28x^4 + 4x^5$$

- Observe  $r(C, x) = r(C_1, x)r(C_2, x)$
- **Why?** Consider  $r_3(C)$ 
  - 3 are in  $C_2$
  - 2 in  $C_2$  and 1 in  $C_1$
  - 2 in  $C_1$  and 1 in  $C_2$



**Figure 8.7**

# Subboards (cont.)

- Generalized: If  $C$  is a chessboard consisting of **pairwise disjoint** subboards  $C_1, C_2, \dots, C_n$ , then

$$r(C, x) = r(C_1, x)r(C_2, x) \cdots r(C_n, x)$$

# Breaking Chessboards

- What if subboards are not disjoint?
- For a square of  $C$ , consider two cases
  - Place a rook on it
  - Do not place a rook on it
- Then, we have  $r_k(C) = r_{k-1}(C_s) + r_k(C_e)$ , which allows us to work on smaller chessboards

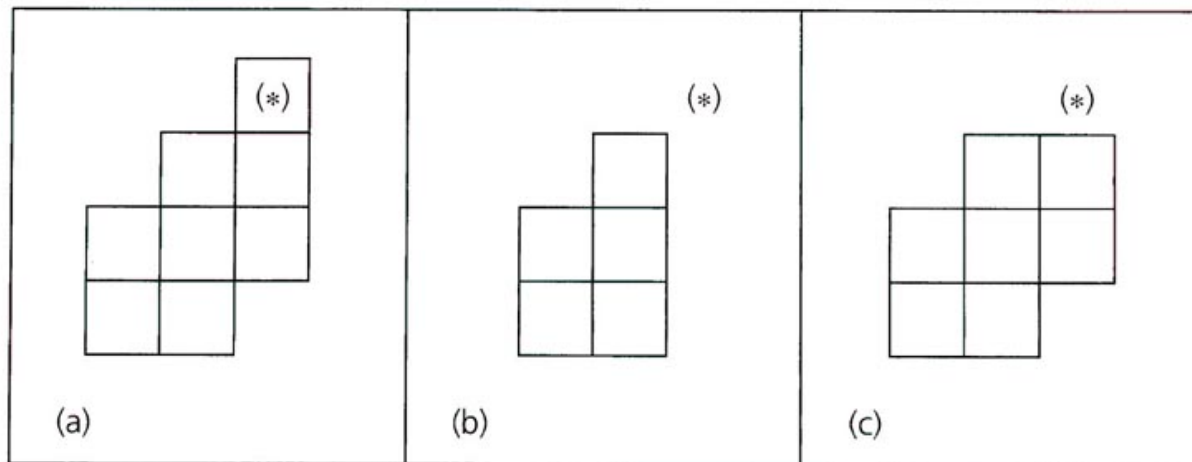


Figure 8.8

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# Application of Rook Polynomial

- Ex 8.15: Four relatives  $R_1, R_2, R_3, R_4$ , who can sit at Table  $T_1, T_2, T_3, T_4$ , but with the following constraints
  - $R_1$  will not sit at  $T_1$  or  $T_2$ ,  $R_2$  will not sit at  $T_2$ ,  $R_3$  will not sit at  $T_3$  or  $T_4$ ,  $R_4$  will not sit at  $T_4$  or  $T_5$
- $c_i$  be the condition,  $R_i$  is in a forbidden (shaded) position.
  - $N(c_1)=N(c_3)=N(c_4)=4!+4!$ , why?
  - $N(c_2)=4!$
  - $N(c_1c_2)=3!$ ,  $N(c_1c_3)=4(3!)$ , why?

	$T_1$	$T_2$	$T_3$	$T_4$	$T_5$
$R_1$					
$R_2$					
$R_3$					
$R_4$					

Figure 8.9

# Application of Rook Polynomial (cont.)

- If we continue, we have

$$S_1 = 7(4!) = 7(5 - 1)!, \quad S_2 = 16(3!) = 16(5 - 2)!$$

- More general,  $S_i = r_i(5 - i)!$ ,  $\forall 0 \leq i \leq 4$ , where  $r_i$  is the number of ways to place nontaking rooks on the shaded squares

- This allows us to use  $r(C, x)$ ,  
to derive  $N(\bar{c}_1 \bar{c}_2 \bar{c}_3 \bar{c}_4)$

	$T_1$	$T_2$	$T_3$	$T_4$	$T_5$
$R_1$					
$R_2$					
$R_3$					
$R_4$					

**Figure 8.9**



# Application of Rook Polynomial (cont.)

- Specifically, using disjoint subboards, we get

$$r(C, x) = 1 + 7x + 16x^2 + 13x^3 + 3x^4$$

- Which leads to  $N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4) = S_0 - S_1 + S_2 - S_3 + S_4$   
 $= 5! - 7(4!) + 16(3!) - 13(2!) + 3(1!) = \sum_{i=0}^4 (-1)^i r_i (5-i)! = 25$

	T <sub>1</sub>	T <sub>2</sub>	T <sub>3</sub>	T <sub>4</sub>	T <sub>5</sub>
R <sub>1</sub>					
R <sub>2</sub>					
R <sub>3</sub>					
R <sub>4</sub>					

Figure 8.9

# Renumbering May Help Calculations

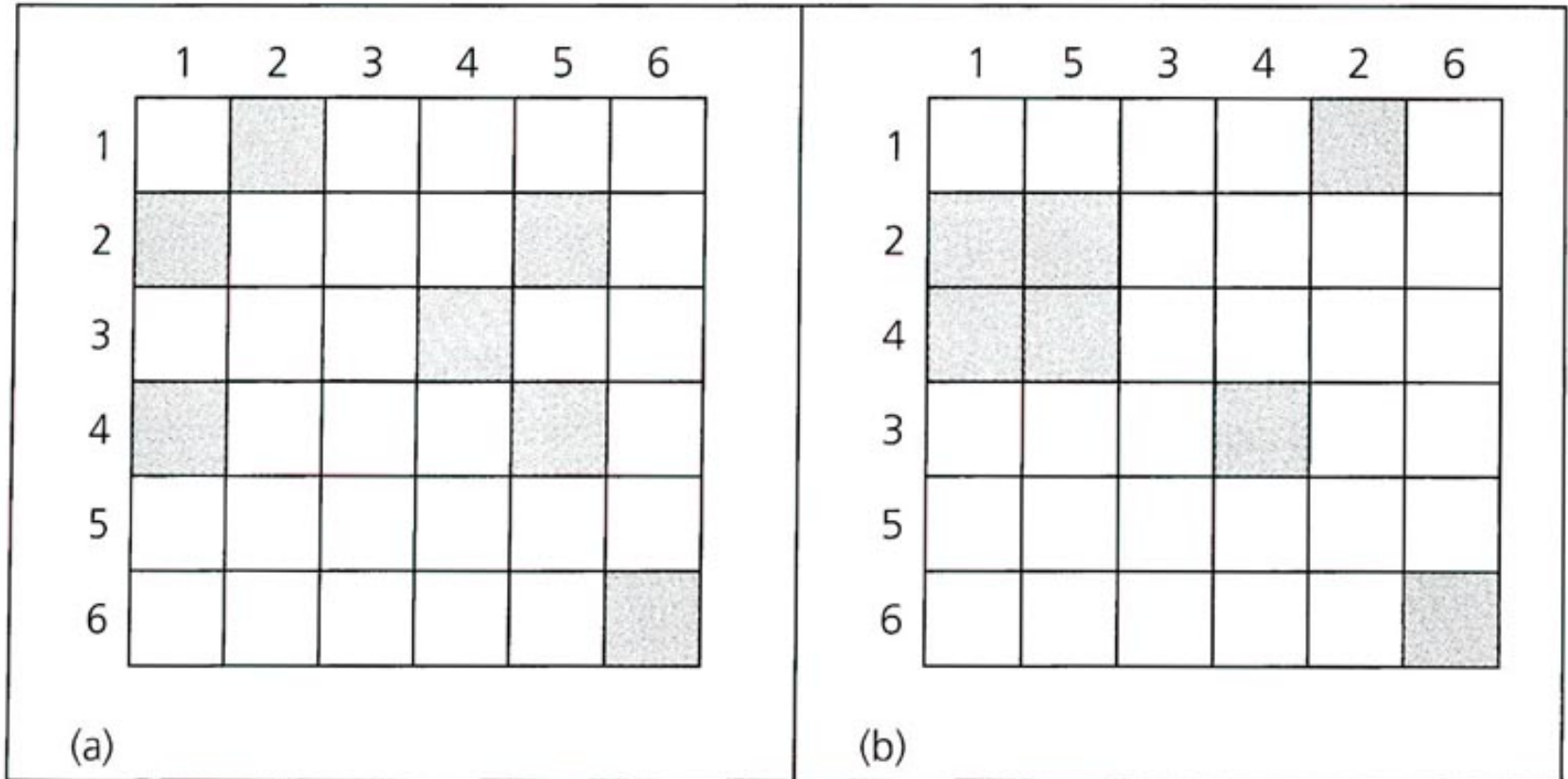


Figure 8.10

# Counting One-to-One Functions

- $A = \{1, 2, 3, 4\}$  and  $B = \{u, v, w, x, y, z\}$ . How many 1-1 functions from  $A$  to  $B$  satisfy none of the following conditions?
  - $c_1: f(1) = u$  or  $v$ ,  $c_2: f(2) = w$ ,  $c_3: f(3) = w$  or  $x$ , and  $c_4: f(4) = x, y, \text{ or } z$
- We are interested in the shaded area

	$u$	$v$	$w$	$x$	$y$	$z$
1						
2						
3						
4						

Figure 8.11

# Counting One-to-One Functions (cont.)

- We have

$$r(C, x) = (1 + 2x)(1 + 6x + 9x^2 + 2x^3) = 1 + 8x + 21x^2 + 20x^3 + 4x^4$$

- Then,

$$N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4) = S_0 - S_1 + S_2 - S_3 + S_4 = \sum_{i=0}^4 (-1)^i r_i \frac{(6-i)!}{2!} = 76$$

- There are 76 1-1 functions with none of the conditions satisfied

	<i>u</i>	<i>v</i>	<i>w</i>	<i>x</i>	<i>y</i>	<i>z</i>
1						
2						
3						
4						

**Figure 8.11**